THE CALCULUS FOR ENGINEERS WITH APPLICATIONS TO TECHNICAL PROBLEMS

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ENGINEERS AND PHYSICISTS:

INTEGRATION AND DIFFERENTIATION, WITH APPLICATIONS TO TECHNICAL PROBLEMS;

AND

CLASSIFIED REFERENCE TABLES

OF INTEGRALS AND METHODS OF INTEGRATION.

BY

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PREFACE TO SECOND EDITION.

The following remarks appeared in the Preface to the First Edition of this book :—"This work aims at the presentation of two leading features in the study and application of the higher mathematics. In the first place, the development of the *rationale* of the subject is based on essentially *concrete* conceptions, and no appeal is made to what may be termed rational imagination extending beyond the limits of man's actual physical and physiological experience. Thus no use is anywhere made of series of infinite numbers of things or of infinitely small quantities. The author believes that the logical development is both sound and complete without reference to these ideas.

"In the second place, a set of Eleven Classified Tables of Integrals and Methods of Integration has been arranged in such manner as seemed best adapted to facilitate rapid reference, and thus relieve the mind engaged in practical mathematical work of the burden of memorising a great mass of formulas.

"Critics who are schoolmen of the pure orthodox mathematical faith may find it hard to reconcile the ideas that have with them become innate, with some of the methods, and possibly some of the phraseology, here adopted. We only ask them to remember that there is arising a rapidly increasing army of men eagerly engaged in the development of physical research and in the industrial applications of scientific results, with the occasional help of mathematical weapons, whose mental faculties have been wholly trained by continuous contact with the hard facts of sentient experience, and who find great difficulty in giving faith to any doctrine which lays its basis outside the limits of their experiential knowledge."

Experience in the use of the book since its first publication has confirmed the author in his belief that the basis upon which its treatment of the Calculus is built is sound, rational, logical, and that its form affords an easy and rapid method of acquiring power to apply, correctly and safely, the higher mathematics to technical problems. The method is good for technicians and physicists because it is easy and rapid. Ease and rapidity would be fundamentally damning faults if it were illogical, or if it did not grow from the roots of the realities of the subject-matter. If it were illogical, it would be destructive of the intellectual training of the student. No illogicality has been discovered in the course of a narrow criticism undertaken during the revision for this second edition. If it be throughout correctly logical, the swifter the habit of logical thinking to which the student is trained, the better for his intellectual growth. But the special virtue of the method is that this intellectual growth in mathematical power is from beginning to end fed by contemplation of the mechanical and physical facts the reality and truth of which are already parts of his familiar mental consciousness; his primary knowledge of which is, indeed, often vague and uncritical, and which he now learns to analyse into strictly definite ideas. If this habit of correlating mathematical thinking with external observation become a confirmed one, then his mental activity, both in logical analysis and in observation of external facts, must automatically develop continuously and permanently after his formal study of mathematics has ceased. It is only by virtue of this habit that mathematical knowledge becomes of use in physics and technical engineering.

The author has no fault to find with the older methods of study of transcendental mathematics, provided always that they be followed only by the select few who by temperament and choice are destined to make pure mathematics their one and only field of lifelong activity. This special kind of activity may be useful to the progress of humanity, and, although the methods are old, they develop year by year in the schools in new directions and arrive at new results. But it is only a very few specially constituted minds which are adapted to pursue these studies successfully.

What needs to be recognised is that it is bad training for the many not so constituted, and—what is of the most urgent importance—that mathematicians of this stamp are unsuited to be, and indeed incapable of being, teachers of technical mathematics.

In the revision for this new edition the work has been very carefully searched for errors. Those that have been discovered, chiefly in the cross-references between Parts I. and II., have been rectified. It is hoped that the volume is now practically free of error.

Considerable additions have been made, mostly in the form of Appendices. These deal for the most part with new applications, the original work of the author, to specially important technical problems, and particularly to the problems of economy in construction. They include, also, additions to Part II. in the Reference Tables of Integrals. In the course of new applications to technical work, general forms of integration which are either new or whose frequent practical utility is novel, demand a place in such Reference Tables. Both in the development of Electrical Engineering and in the stricter application of scientific method to Mechanical Design, this process of development is almost continuous and inevitable.

ROBERT H. SMITH.

1908.

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THE CALCULUS FOR ENGINEERS.

CHAPTER I.

INTRODUCTORY.

1. Integration more useful than Differentiation.—In physical and engineering investigations the Integral Calculus lends more frequent aid than does the Differential Calculus, and the problems involving the Integral are more often of a practically important type than those requiring the Differential Calculus alone in their solution. But the ordinary student of mathematics never reaches even an elementary knowledge of Integration until he has mastered all but the most recondite portions of the science of Differentiation.

It seems *a priori* irrational, and contrary to a liberal conception of educational policy, to teach the higher mathematics in a manner so contrary to almost self-evident utility. Adherence to this the orthodox method of teaching in the Schools and Universities is, no doubt, responsible for the persistent unpopularity of this branch of knowledge and intellectual training among the classes devoted to practical work.

2. Method of the Schools.—It must be admitted that no great progress can be made in Integration without help from the results obtained by Differentiation. Therefore, so long as the two are taught as distinct subjects, by the aid of separate text-books, it is a distinct convenience to the teachers to finish off one before entering upon the other. If they be thus separated into two successive periods of study, it becomes a practical necessity to give Differentiation the priority in point of time.

3. Rational Method.—Still, it by no means follows that the whole of the science of Differentiation must be known before any of that of Integration can be explained, thoroughly mastered, and utilised. The ordinary system of teaching the subject forces the practical student to spend on Differentiation an amount of time altogether needless for his professional objects before he enters upon Integration. Much of the former he will never use. The latter, from the very beginning, will supply him with abundant problems of immediate interest and importance in his own special work, and will, moreover, furnish him with a powerful engine that will enormously lighten the difficulties of his own professional subjects and make his progress in these tenfold more rapid.

Let it be noted, also, that very frequently the reasoning used to find an integral is essentially the same as that used to find the inverse differential. It is thus pure waste of time to go through this reasoning twice over. Once understood, it leads to the *simultaneous* recognition of the two inverse results, both of them, it may be, eminently useful. Therefore, as far as practicable, the study of Differentiation and Integration ought to be pursued *pari passu.*

4. **Active Interest in the Study.**—In modern education, in which such large demands are made upon the intellectual energies of the pupil, the necessity of the stimulus of a real active interest, opening out easily recognised prospects of broadening and deepening knowledge and of utilitarian advantage, ought to be conceded in the freest and most liberal fashion. Moreover, it is right to lead the pupil along the easiest road, provided it be a legitimate one. The thoroughness of the training he receives in habits of sound, trustworthy scientific thought depends more upon the length of time he is guided within the limits of correct method, and less upon whether he travels a short distance on a rugged and difficult path or a long distance upon a plainer and smoother route.

5. **Object of Present Treatise.**—The object of the present treatise is to introduce the student at once to the fundamental and important uses of the Integral Calculus, and incidentally to those of much of the Differential Calculus. This we desire to do in such a way as to stimulate a growing desire to progress always further in a branch of science which soon shows itself capable of supplying the key to so many practical investigations.

6. **Clumsiness of Common Modes of Engineering Analysis.**—At the present time our technical text-books are loaded with tedious and clumsy demonstrations of results that can be obtained "in the twinkling of an eye" by one who has grasped even only the elements of the Calculus. These demonstrations are supposed to be "elementary." They are not really so; each of them really contains, hidden with more or less skill, identically the same reasoning as that employed in establishing the Calculus formulas applicable to the case in hand. They are, in fact, simply laboured

methods of cheating the student into using the Calculus without his knowing that he is so doing. There is no good reason for this. The elements of the Calculus may be made as easy as those of Algebra or of Trigonometry. More good, useful scientific result can be obtained with less labour by the study of the Calculus than by that of any other branch of mathematics.

7. **Graphic Method.**—Much of the Calculus can be rigorously proved by the graphic method; that is, by diagram. This method is here used wherever it offers the simplest and plainest proof; but where other methods seem easier and shorter they are preferred. The present book is strictly confined to its own subject; and, wherever it is necessary, the results proved in books on Geometry, Algebra, Trigonometry, etc., are freely made use of; employing always, however, the most elementary and most generally known of these results as may be sufficient for the purpose.

8. **Illustrations.**—Everywhere the meaning and the utility of the results obtained are illustrated by applications to mechanics, thermodynamics, electrodynamics, problems in engineering design, etc., etc.

9. **Classified List of Integrals.**—The part of the book which is looked upon by its authors as the most important and the most novel is the last, namely, the Classified Reference List of Integrals. This is really a development of a Classified List of Integrals which one of the authors made twenty years ago to assist him in his theoretical investigations, and which he has found to be continuously of very great service. He has never believed in the policy of a practical man's burdening his memory with a load of theoretical formulas. Let him make sure of the correctness of these results, and of the methods by which they were reached. Let him very thoroughly understand their general meaning, and especially the limits of their range of applicability; let him recognise clearly the sort of problem towards the solution of which they are suited to help; let him practise their application to this sort of problem to an extent sufficient to make him feel sure of himself in using them in the future in the proper way. Then let him keep notes of these results in such a manner as will enable him to find them when wanted without loss of time; and let him particularly avoid wasting his brain-power by preserving them in his memory. The more brain-power is spent in memorising, the less is there left for active service in vigorous and wary application in new fields to attain new results. Formulas have a lamentable characteristic in the facility they offer for wrong application. A formula fixed perfectly in the memory, and the exact meaning and correct mode and limits of whose application are imperfectly understood, is a

pure source of misfortune to him who remembers it. It is infinitely more important to cultivate the faculty of cautious and yet ready use of formulas than to have the whole range of mathematical formulas at one's finger ends; and this is also of immensely greater importance to the practical man than to keep in mind the proofs of the formulas.

To obviate the necessity of such memorisation the "Classified Reference List of Integrals" has been constructed in the manner thought most likely to facilitate the rapid finding of whatever may be sought for. The results are not tabulated in "rational" order, that is, in the order in which one may be logically deduced from preceding ones. They are classified, firstly, according to subject, *e.g.*, Algebraical, Trigonometrical, etc., etc., and under each subject they are arranged in the order of simplicity and of most frequent utility. A somewhat detailed classification has been found desirable in order to facilitate cross-references, the free use of which greatly diminishes the bulk of the whole list. The shorter such a list is made, the easier is it to make use of.

10. Scope of Prefatory Treatise.—This treatise does not prove all the results tabulated in the "Reference List." The latter has been made as complete as was consistent with moderate bulk, and includes all that is needed for what may be described as ordinary work, that is, excluding such higher difficult work as is never attempted by engineers or by undergraduate students of physics. The treatise aims at giving a very thorough understanding of the principles and methods employed in finding the results stated concisely in the "Reference List"; proofs of all the fundamentally important results; and, above all, familiarity with the practical uses of these results, so as to give the student confidence in his own independent powers of putting them to practical use. The last chapter on the Integration of Differential Equations ought to aid greatly in pointing out the methods of dealing with various classes of problems. The ninth chapter, on Maxima and Minima, is perhaps more illustrative than any other of the great variety of very important practical problems that can be solved correctly only by the aid of the Calculus.

CHAPTER II.

GENERAL IDEAS AND PRINCIPLES—ALGEBRAIC AND GRAPHIC SYMBOLISM.

11. Meaning of a "Function."—Suppose that a section be made through a hilly bit of country for some engineering purpose, such as the making of a highway, or a railway, or a canal. The levels of the different points along the section are obtained by the use of the Engineer's Level, and the horizontal distances by one or other of the ordinary surveying methods. Let fig. 1 be the plotting on paper of the section. According to ordinary practice, the heights would be plotted to a much opener scale than the horizontal distances; but in order to avoid complication in a first illustration, we will assume that in fig. 1 heights and distances are plotted to the same scale.

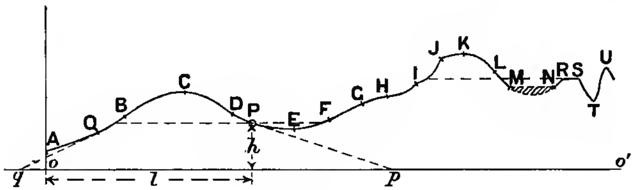

Fig. 1.

Each point P on this section is defined strictly by its level h and its horizontal position l. The former is measured from some conveniently chosen datum level. The latter is measured from any convenient starting-point. These two are called by mathematicians the **co-ordinates** of the point P on the curve ABC, etc.

For each ordinate l there is one defined value of the co-ordinate h; except throughout the stretch MN, where a break in the curve occurs. Putting aside this exception, the height h is, when this strictly definite relation exists, called in mathematical language a "**function**" of l; or

$$\text{Height} = \text{Function of Horizontal Distance},$$

or, more simply written in mathematical shorthand,

$$h = \mathrm{F}(l).$$

12. Ambiguous Cases.—As seen from the dotted line drawn horizontally through P, there are three points on the section at the same level. Thus the statement that

$$\text{Distance} = \text{Function of Height}$$

or

$$l = f(h)$$

must be understood in a somewhat different sense from the first equation: namely, in the sense that, although for each height there correspond particular and exactly defined distances, still two or more such distances correspond to one and the same height, so that, if nothing but the height of a point were given, it would remain doubtful which of two or three horizontal positions it occupied. This ambiguity can only be cleared away by supplying special information concerning the point beyond that contained in the equation.

13. Inverse Functions.—The two formulas

$$h = F(l)$$

and

$$l = f(h)$$

are simply two different forms into which the relation between h and l, or the **equation to the curve**, can be thrown. The first form may be called the **solution of the equation for** h; the second the **solution of the equation for** l.

The functions F( ) and f( ) are said each to be the "inverse" of the other. An inverse function is frequently indicated by the symbol -1 put in the place of an index. For example, if s be $\sin \alpha$, then the angle α may be written $\sin^{-1}s$. Or if l be the logarithm of a number N, or $l = \log N$; then $N = \log^{-1}l$, which expression means that "N is the number whose log is l."

14. Indefiniteness of a Function in Special Cases.—The stretch of ground from R to S is level. Here the value of h corresponds to a continuously varying range of values of l. For this particular value of h, therefore, we have *between certain limits indefiniteness* in the solution for l.

If there were under the point J a stretch of perfectly vertical cliff, then for the one value of l to this cliff the solution for h would be similarly *indefinite* between the limits of level at the foot and at the top of the cliff.

15. Discontinuity.—From M to N there is a break in the curve. In such a case mathematicians say that h is a **discontinuous** function of l; the discontinuity ranging from M to N.

16. Maxima and Minima.—From A to C the ground rises; from C to E it falls. At C we have a summit, or a **maximum** value of h. This maximum *necessarily* comes at the end of a rising and the beginning of a falling part of the section. Evidently the converse is also true, viz., that **after** a rising and **before** the following falling part there is *necessarily* a maximum, *provided there be no discontinuity between these two parts.* There is another maximum or summit at K.

The ground falls continuously from C to E, and then rises again from E onwards. There is no discontinuity here, and E gives, therefore, a lowest or **minimum** value of h. This necessarily comes **after** a falling and **before** a rising part of the section; and between such parts there necessarily occurs a minimum, if there be no discontinuity.

We have here assumed the forward direction along the section to be from A towards the right hand. But it is indifferent whether we call this or the reverse the forward direction as regards the distinction between maximum and minimum points.

17. Gradient or Differential Coefficient.—Each small length of the section has a definite slope or **gradient.** Engineers always take as the measure of the gradient the ratio of the rise of the ground between two points near each other to the horizontal distance between the same points. This must be carefully distinguished from the ratio of the rise to distance measured along the sloping surface. This latter is the *sine* of the angle of inclination of the surface to the horizontal; whereas the gradient is the *tangent* of the same angle of inclination. This gradient is the rate at which h increases with l. It is, in the present case, what is called a **space** rate, or length rate, or linear rate, because the increase of h is compared with the increase of a length l (not because h is a length, but because l is a length).

If at the point Q the dotted line Qq be drawn touching the section curve at Q, then the gradient at Q is the tangent of the angle QqO. The touching line at point P on the downward slope cuts OO′ at p, and the tangent of PpO′ is negative. It equals the tangent of PpO with sign reversed.

In the language of the Calculus this gradient is called the **Differential Coefficient of h with respect to l.** Taking the forward direction as from A towards the right hand, the gradient is **upward** or **positive** from A to the summit C; **downward** or **negative** from C to the minimum point E; positive again from E to K, and negative from K to M. From N to R it is positive, and along RS it is zero.

18. Gradients at Maxima and Minima.—At each maximum and minimum point (C, E, K) the gradient is zero. At each maximum point (C, K) it passes through zero from positive to negative. At a minimum point (E) it passes through zero from negative to positive.

At H there is also level ground, or zero gradient. Here, however, there is neither maximum nor minimum value of h. This point comes between two rising parts of the section: there is a positive gradient both before and after it. Although, therefore, we find zero gradient at every maximum and at every minimum point, it is not true that we necessarily find either a maximum or a minimum wherever there is zero gradient.

19. Change of Gradient.—On the rising part of the ground it becomes gradually steeper from A to B; that is, the upward gradient increases. Otherwise expressed, there is a positive increase of gradient. From B to C, however, the steepness decreases; there is a decrease of gradient, or the variation of gradient is negative (the gradient itself being still positive).

Thus the variation of gradient being positive from A to B and negative from B to C, passes through the value zero at B, the point where the gradient itself is a maximum.

From C to D the gradient is negative, and becomes gradually steeper; that is, its negative value increases, or, otherwise expressed, its variation is negative. From D to E the gradient is negative, but its negative value is decreasing, that is, its variation is positive. Thus at D the variation, or rate of change of gradient, changes from negative to positive by passing through zero, and at this point D we have the steepest negative gradient on this whole slope CE. The steepest negative gradient, of course, means its *lowest* value. Thus at D we have a minimum value of the gradient.

20. Zero Gradients.—The distinction between the three parts C, E, and H, at all of which the gradient is zero, becomes now clear. At C the variation of the gradient is **negative**, and this gives a **maximum** height. At E this variation is **positive**, and here there is a **minimum** height. At H this rate of variation of the gradient is zero, and here, although the gradient be zero, there is neither maximum nor minimum height.

21. Discontinuity or Break of Gradient.—Wherever there is a sharp corner in the outline of the section, as at I, J, R, S, T, U, there is a **sudden** change or break of gradient. This means that at each of these points there is **discontinuity of gradient**; and the above laws will not apply to such points.

Wherever there are points of discontinuity, either in the curve itself or in its gradient, special methods must be adopted in any investigations that may be undertaken in regard to the characteristics of the law connecting the ordinates. The methods applicable to the continuous parts of the curve may, and usually do, give erroneous results if applied to discontinuous points.

22. Infinite Gradient.—Under J the face being vertical, the gradient is commonly said to be "**infinite.**" At each of the sharp points I, J, R, S, T, U, the variation of gradient being sudden, the rate of variation of gradient becomes "infinite." More correctly expressed, there exists no gradient at J; and at I, J, R, etc., there are no rates of variation of gradient.

23. Meaning of a "Function."—The symbolic statement

$$h = F(l)$$

is not intended to assert that the relation between l and h is expressible by any already investigated mathematical formula, whether simple or complicated. For example, in fig. 1 the said relation would be extremely difficult to express by any algebraic or trigonometric formula. Equally complicated would be the law expressing the continuous variation of, for example, the horse-power of a steam-engine on, say, a week's intermittent running; or that connecting the electric out-put of a dynamo when connected on to a circuit of variable and, perhaps, intermittent conductivity. Yet separate short ranges of these laws may in many cases be approximated to by known mathematical methods; and even when this is not possible, many very interesting, important, and practically useful special features of the general law may be investigated by mathematical means, without any exact knowledge of the full and complete law. Thus without making any reference to, or any use of, the exact form of the function F( ) in the equation applicable to fig. 1, we have already been able to point out many important features of the law it represents.

24. Horse-power as a Function of Pressure.—Again, although the actual running of, say, a steam-engine from minute to minute varies with many changes of condition, still, if we choose to investigate the separate effect of one only of these changes, for instance, change of initial pressure, it may be found fairly simple. Thus we may write

$$\text{Horse-power} = \text{Function of Initial Pressure},$$

or

$$\text{HP} = \phi(p),$$

where p is the pressure. This means that any change of pressure changes the horse-power; and to investigate the separate effect of change of pressure on horse-power, we consider all the other conditions to remain (if possible) constant, while the pressure changes. Some other conditions may themselves necessarily depend on the pressure, and these, of course, cannot be assumed to remain constant. For example, the cut-off may be supposed to remain constant. But the amount of initial steam condensation in the cylinder depends partly on the initial pressure, and it cannot, therefore, be assumed a constant in the equation HP $=\phi(p)$. Similarly, the HP may be considered as a function of the speed, it alone being varied while all other things are kept constant. Or the HP may be taken as a function of the cut-off, the initial pressure, the speed, and everything else being kept constant, while the cut-off is varied.

25. Function Symbols.—When different laws connecting certain varying quantities have to be considered at the same time, different symbols, such as

$$F(l),\ f(l),\ \phi(l),\ \psi(l),$$

are used to indicate the different **functions** of l referred to.

26. Choice of Letter-Symbols.—In fig. 1 we have used l to represent a distance, because it is the first letter in the word "*length*," and similarly h to represent "*height*." It is very desirable when letter-symbols have to be used, to use such as readily call to mind the nature of the thing symbolised. Especially in practical applications of mathematics, and more particularly when there is any degree of complication in the expressions involved, is the adherence to this rule to be strongly recommended. By keeping the mind alive to the nature of the things being dealt with, error is safeguarded against, and the true physical meaning of the mathematical operations and of their results are more easily grasped. Without a complete understanding of the physical meaning of the result, not only is the result useless to the practical man, but its correctness cannot be judged of. If, on the other hand, the physical meaning be fully grasped, any possible error that may have crept in in the mathematical process of finding the result, is likely to be detected and its source discovered without great difficulty.

27. Particular and General Symbols.—But many mathematical rules and processes have such wide application to so many entirely different physical conditions, that, in order the more clearly to demonstrate the generality of their application, mathematicians prefer to use letter-symbols chosen purposely so as to suggest only with difficulty anything endowed with special characteristics; such as x, y, z, symbols which do not suggest to the mind any idea whatever except that of absolute blankness.

It is doubtful whether this is a desirable habit in mathematical training. It seems probable that a course of reasoning might be more firmly established in the mind of the student if he were first led through it in its concrete and particular aspect—the mind being kept riveted on one special set of concrete meanings to be attached to his symbols—and then afterwards, if need be, he may go through it again once, twice, or, if necessary, a dozen times, in order to discover (if or when this be true) that the **general form** of the result will remain the same whatever particular concrete meanings be attached to his symbols.

28. x, y, **and** z.—There is one feature in the use ordinarily made of x, y, z in mathematical books which the writer thinks is a real evil. In his earlier chapters the orthodox mathematician establishes a habit of using y to indicate a *function of* x: he constantly writes $y = f(x)$: that is, he takes y to represent a thing dependent on x, and which necessarily changes in quantitative value when x changes. But in his later chapters he uses y and x as two **independent variables**, that is, as two quantities having no sort of mutual dependence on each other, the variation of either one of which has no effect whatever upon the other. This is apt to, and does, produce confusion of mind; especially as regards the true meaning of different sets of formulas very similar in appearance, one referring to y and x as mutually dependent quantities, the other referring to y and x as independent variables.

29. Functions of x.—When x is used to indicate a variable quantity, any other quantity whose value varies in a definite way with the varying values of x, may be symbolically represented in any of the following ways:—

$$F(x), f(x), \phi(x), \psi(x), \chi(x), \text{ and } X, \mathfrak{X} \text{ or } \Xi.$$

The last forms, X, etc., are for shortness and compactness as convenient as y, and are more expressive. They will be used chiefly in connection with x in the following pages.

30. X dependent on x.—X may mean a function which is capable of being also changed by changing the values of one or more other quantities besides x; but in so far as it is considered as a function of x, consideration of these other *possible* changes is eliminated by supposing them not to occur. This is legitimate because these other elements which go to the building up of X do not *necessarily* change with x. All elements involved in X, which necessarily change with x, are to be expressed in terms of x, and their variation is thus taken account of in calculating the variation of X.

31. Nature of Derived Functions.—In dealing with functions of this kind, mathematicians call x the "**independent variable**," a somewhat unhappy nomenclature. X and x are in physical reality mutually dependent one on the other. In the mathematical formula, however, X being expressed in terms of x, it is considered as being derived from, or dependent upon, x; the various values of X being calculated from those of x, and the changes in X being calculated from the changes in x. Thus it should be borne in mind that the dependence of the one on the other suggested in the commonly used phrase "independent variable" is purely a matter of method of calculation, and not one of physical reality.

32. Variation of a Function.—Similarly Y may be used to indicate a derived or "**dependent**" function whose value depends only upon constants and upon the variable y.

Or L may be made to denote a derived function depending only on constants and on the variable l.

33. Scales for Graphic Symbolism.—Those readers of this treatise who are engineers must, from practice of the art of Graphic Calculation, be familiar with the device of representing quantities of all kinds by the **lengths of lines** drawn upon paper, these lengths being plotted and measured to a suitable **Scale.**

So long as the **quantity** of a function is its only characteristic with which we are concerned, each quantity can always be represented by the length of a line drawn in any position and in any direction on a sheet of paper, the scale being such that 1 inch or 1 millimetre of length represents a convenient number of units of the kind to be represented. In "Graphic Calculation" we very commonly represent on the paper also the two other characteristics of **position** and **direction** of the things dealt with; but in the Differential and Integral Calculus, so far as it is dealt with in this treatise, we are concerned *only with quantity.*

It is convenient to draw all lines representing the various values of the same kind of thing in one direction on the paper. Thus we may plot off all the x's horizontally and the corresponding X's vertically. If, when the magnitude of x is varied continuously (*i.e.*, without break or gap), the change of X be also gradual and continuous, there is obtained by this process a continuous curve which is a complete graphic representation of the law connecting X and x. The student ought at the outset to understand fully the nature of this kind of representation. It is clear that it is in its essence as wholly conventional and symbolic as is the letter-symbolism of ordinary algebra. Spoken words, written words, and written numbers are in the same way conventional; they also constitute systems of arbitrary symbolism. Graphic diagram representation is neither more nor less symbolic and arbitrary than ordinary language.

34. Ratios in Graphic Delineation.—The curve in fig. 2 is such a graphic delineation of a law of mutual dependence between X and x. If X be of a different kind from x, it is impossible to form any numerical ratio between the two scales to which X and x are plotted. For instance, if the X's are ft.-lbs. and the x's feet, then the vertical scale may be, perhaps, 1″ = 10,000 ft.-lbs., and the horizontal scale 1″ = 10 ft.; but the number $\frac{10,000}{10}$, or 1000, is not a pure ratio between the two scales. But in physics we have relations between things of different kinds, which are called **physical ratios.** It is only by use of such physical ratios that **derived**

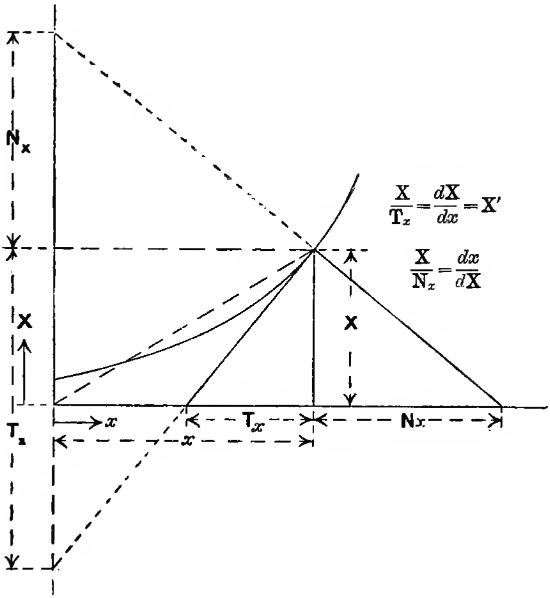

FIG. 2.

quantities are obtained. Thus the physical ratio between a number of ft.-lbs. and a number of feet, or ft.-lbs./ft., is a number of lbs. The ratio is of an altogether different **kind**, in this case lbs., from either that of the dividend or that of the divisor.

Now the ratio between a height and a horizontal distance on the paper is a gradient measured from the horizontal.

In this example, then, a gradient would mean a number of lbs., and each gradient would represent lbs. to a certain scale. Continuing the above example, a tangent or gradient measuring unity on the paper, *i.e.*, the tangent of 45°, would mean

$$\frac{10{,}000 \text{ ft.-lbs.}}{10 \text{ ft.}} = 1000 \text{ lbs.}$$

This is the scale to which gradients from axis of x are to be measured; or

$$\text{Unit gradient} = \frac{\text{Unit height on paper}}{\text{Unit horizontal length on paper}} = 1000 \text{ lbs.}$$

Gradients measured from the axis of X have a reciprocal interpretation and are to be measured to a reciprocal scale. Thus x/X means ft./ft.-lbs., or 1/lbs., that is, the reciprocal of a number of lbs.

35. Differential Coefficient, x-Gradient or X′.—The gradient from axis of x of the curve at any point x, X, is called the "**Differential Coefficient** of X **with respect to** x," and is symbolised by

$$\text{either } \frac{dX}{dx} \text{ or } X'.$$

The phrase "Differential Coefficient of X with respect to x" is a cumbrous one. A shorter phrase is

the x-gradient of X;

and as this phrase is very easily understood and definitely descriptive, it is used in this treatise.

The gradient of the curve from the axis of X is the reciprocal of the above. It is called

the Differential Coefficient of x **with respect to** X, or
the X-gradient of x;

and is shortly written $\frac{dx}{dX}$.

36. Scale of X′.—The scales of X and x are in general different; and that of X′ must always be different from either of these. The numerical relation between these scales and that of $\frac{dx}{dX}$ may be thus expressed:

Let

the scale of the x's be 1″ = s units of the x kind or quality;
" " X's " 1″ = S " " X " "

then

the scale of the X″s is

$$\text{Unit gradient} = \tan 45° = \frac{S}{s} \text{ units of the } \frac{X}{x} \text{ kind};$$

and

the scale of the $\frac{dx}{dX}$ is

$$\text{Unit gradient} = \tan 45° = \frac{s}{S} \text{ units of the } \frac{x}{X} \text{ kind.}$$

37. Sign of X′.—The sign of X′ is + when the slope of the curve is such as to make both X and x increase positively at the

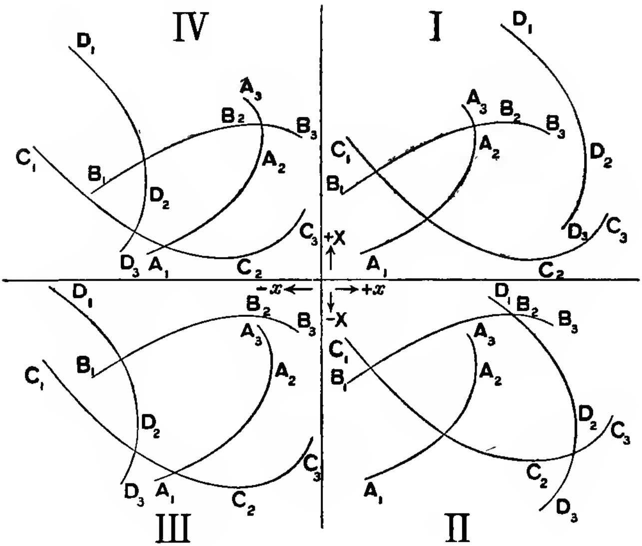

Fig. 3.

same time; it is − when it makes one increase while the other decreases. Evidently $\frac{dx}{dX}$ must always have the same sign as X′. The possible variations of X′ and $\frac{dx}{dX}$ are very fully illustrated in fig. 3.

In fig. 3, $+x$ is measured towards the right and $+X$ upwards; negative x's are measured towards the left and negative X's downwards.

The student should follow out the variations from + through 0 to − of both X′ and $\frac{dx}{dX}$ throughout the lengths of all the four curves A, B, C, and D in each of the four quadrants.

38. Subtangent and Subnormal.—In fig. 2 there are drawn three lines from a point xX of the curve, viz., a vertical, a tangent, and a normal. These intercept on the axis of x the lengths marked T_x and N_x on the diagram. T_x is called the subtangent and N_x the subnormal.

Since (by definition) the tangent has the same gradient as the curve at its touching point, evidently

$$X' = \frac{X}{T_x}, \text{ or } T_x = \frac{X}{X'}$$

and

$$\frac{dx}{dX} = \frac{1}{X'} = \frac{X}{N_x}, \text{ or } N_x = X'X.$$

Here T_x measured to the x-scale, and interpreted as being of the same **kind** as the x's, is a true graphic representation of X/X'.

But $\frac{dx}{dX}$ is of the same **kind** as x/X, and, therefore, would be **not** of the same kind as X/N_x, if N_x were measured to the x-scale, and interpreted as of the same **kind** as x. Thus in order that N_x may be used as a true graphic representation of $X'X$, which is of the same **kind** as X^2/x, care must be taken **not** to measure it to the x-scale, and **not** to interpret it as the same kind of thing as x.

If the diagram were replotted, leaving the x-scale unaltered, and making the X-scale more open, the paper-height of X would be increased, and the paper-gradient X′ would be increased in the same proportion. It can easily be shown that the paper-length of T_x would remain unaltered, while that of N_x would be increased in a ratio which is the square of that in which X is increased. Similarly if, while the X-scale is maintained unchanged, the x-scale were altered so as to increase the paper-length of x, then the paper-gradient of the curve would be decreased in the same proportion as x is increased; T_x would be increased in the same proportion as x; N_x would be decreased in the same proportion.

Thus N_x in order to be a true graphic representation of $X'X$, a quantity whose dimensions are those of X^2/x, must be measured to the scale

$$1'' = \frac{S^2}{s} \text{ units of the } \left(\frac{X^2}{x}\right) \text{ kind.}$$

In fig. 2, T_x and N_x are taken upon the x-axis, and may be termed the x-subtangent and x-subnormal. If the curve-touching line and the normal be prolonged to cut the X-axis, they and the horizontal through the touching-point will give intercepts on the X-axis, which may be termed the X-subtangent and X-subnormal, and may be written T_X and N_X. They are shown on fig. 2, and their proper scales are given below.

39. Scale of Diagram Areas.—An area enclosed by any set of lines upon such a diagram may be taken as the graphic representation of a quantity of the same kind as the product Xx, and must be measured to the scale, 1 sq. inch = (Ss) units of the (Xx) kind.

40. Table of Scales.—The following is a table of interpretations of the diagram. This diagram will be constantly used hereafter for both illustrations and proofs, which latter cannot be accepted as legitimate unless the whole nature of this manner of symbolic expression be intimately understood.

TABLE OF INTERPRETATIONS AND SCALES OF DIAGRAMMATIC OR GRAPHIC REPRESENTATIONS OF DERIVATIVE AND DERIVED FUNCTIONS.

Name.	Interpretation.	Symbol.	Diagram Scale.	
Variable, ...	x	x	$1'' = s$	units of x kind.
Function of x, . .	X	X	$1'' = S$	,, X ,,
x-Gradient of X, .	Ratio of small increase of X to accompanying increase of x.	X′	$\tan 45° = \frac{S}{s}$	,, X/x ,,
X-Gradient of x, .	Ratio of small increase of x to accompanying increase of X.	$\frac{dx}{dX}$	$\tan 45° = \frac{s}{S}$	,, x/X ,,
x-Subtangent, . .	X/X'	T_x	$1'' = s$	,, x ,,
x-Subnormal, .	XX'	N_x	$1'' = \frac{S^2}{s}$	,, X^2/x ,,
X-Subtangent, .	$x\Big/\frac{dx}{dX} = xX'$	T_X	$1'' = S$	,, X ,,
X-Subnormal, . .	$x\Big/X' = x\frac{dx}{dX}$	N_X	$1'' = \frac{s^2}{S}$	,, x^2/X ,,
Area, ...	xX	A	1 sq. in. $= sS$	,, xX ,,

41. Increments.—In going forward from a point on the curve a little way, a rise occurs if the gradient be upwards. The short distance measured along the sloping curve may be resolved into two parts, one parallel to axis of x, the other parallel to axis of X. These two parts are the projections of the sloping length upon the two axes. They constitute the **differences** of the pairs of x and X co-ordinates at the beginning and the end of the short sloping length. These differences are designated by the Greek δ; thus, see fig. 4,

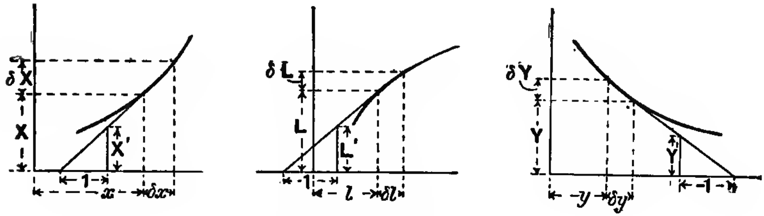

FIG. 4.

δx projection on x-axis, and
δX „ „ X „

Since the gradient X′ is the ratio of rise to horizontal distance throughout a short length, it is evident that

$$\delta X = X'\delta x.$$

If l and L be the co-ordinates, and if the gradient be called L′, then this would be written

$$\delta L = L'\delta l.$$

If y and Y were the co-ordinates, the gradient being called Y′, then

$$\delta Y = Y'\delta y.$$

42. Increment in Infinite Gradient.—These are the direct self-evident results of the definition of gradient, or differential co-efficient. They do not, of course, apply to points where there is no gradient, that is, to sharp corners in a diagram, where the direction of the diagram line changes **abruptly**.

If the diagram line run exactly vertical at any part, then for that part X′ becomes **infinite**, and the equation appears in the form

$$\delta X = \infty \times 0$$

an indeterminate form.

This last case corresponds to the piece of vertical cliff under point J in the section fig. 1.

43. Integration.—The general case corresponds to the gradual stepping along the other parts of this section. The length of each step is projected horizontally (δl or δx) and vertically (δh or δX). The latter is the rise in level, and it equals the gradient multiplied by the horizontal projection of the length of step.

In stepping continuously from one particular point on the section to another, for instance, from A to C on fig. 1, the total horizontal distance between the two is the sum of the horizontal projections (the δl's or δx's) of all the separate steps; and the total difference of level is the sum of the vertical projections (the δh's, or δL's, or δX's) of all the separate steps. In climbing the hill, the climber rises the whole difference of level from A to C, step by step: the total ascent is the sum of all the small ascents made in all the long series of steps. If the distance be considerable, the number of steps cannot be counted, except by some counting instrument, such as a pedometer; but the total ascent remains the same, whether it be accomplished in an enormous number of extremely short steps or in an only moderately large number of long strides.

The mathematical process of calculating these sums is called **Integration.**

This mathematical process is indicated by the symbol the Greek capital Σ, when the individual steps are of definitely measurable small size. But when the method of summation employed is such that it assumes the steps to be minutely and immeasurably small, the number of them being proportionately immeasurably large, and when, therefore, of necessity the method takes no account of, and is wholly independent of, the particular minute size given to the steps, then the symbol employed is $\int$, which may be looked upon as a specialised form of the English capital S, the first letter of the word "sum." The result of the summation is called the **Integral.**

44. Increment Symbols.—The separate small portions, whose sum equals the Integral, are called the **Increments** or the **Differentials.**

When the increments are of definitely measurable small size, they are indicated by the symbols δx, δX, δh, δL, δY, etc., etc.

When they are immeasurably minute, and their number correspondingly immeasurably large, they are indicated by the symbols dx, dX, dh, dL, dY, etc., etc.

45. Integration Symbols. Limits of Integration.—The integration is carried out between particular **limits,** such as B and C in fig. 1. These limits are sometimes written in connection with the symbols of integration, thus:

$$\sum_{\mathrm{B}}^{\mathrm{C}} \delta h\,,\ \sum_{\mathrm{B}}^{\mathrm{C}} \delta l \text{ or } \int_{\mathrm{B}}^{\mathrm{C}} dh\,,\ \int_{\mathrm{B}}^{\mathrm{C}} dl\,.$$

If $l_{\mathrm{C}} h_{\mathrm{C}}$ be the co-ordinates of the point C, fig. 1, and $l_{\mathrm{B}} h_{\mathrm{B}}$ be those of the point B, then these integrals mean the same thing as

$$(h_{\mathrm{C}} - h_{\mathrm{B}}) \text{ or } (l_{\mathrm{C}} - l_{\mathrm{B}})\,.$$

The limits are above indicated in the symbol by the names only of the points referred to. The points themselves are, however, frequently indicated only by the values of their co-ordinates, and then it is customary to indicate the limits of integration by writing at top and bottom of the sign of integration the limit-values of the variable whose increment appears in the intregral. Thus, since

$$\delta h = \frac{dh}{dl}\delta l$$

we have the integral of δh between B and C expressible in the two following forms:

$$h_{\mathrm{C}} - h_{\mathrm{B}} = \sum_{h_{\mathrm{B}}}^{h_{\mathrm{C}}} \delta h = \sum_{l_{\mathrm{B}}}^{l_{\mathrm{C}}} \frac{dh}{dl}\delta l = \int_{h_{\mathrm{B}}}^{h_{\mathrm{C}}} dh = \int_{l_{\mathrm{B}}}^{l_{\mathrm{C}}} \frac{dh}{dl} dl\,.$$

If particular points be indicated by numbers, the symbolism becomes somewhat neater. Thus the integral of $\delta \mathrm{X}$ between the points 1 and 2 of the x, X curve at which points the ordinates may be called $x_1\ x_2$, and the co-ordinates $\mathrm{X}_1\ \mathrm{X}_2$, is

$$\mathrm{X}_2 - \mathrm{X}_1 = \int_{\mathrm{X}_1}^{\mathrm{X}_2} d\mathrm{X} = \int_{x_1}^{x_2} \mathrm{X}' dx\,.$$

Or, again, if it were convenient to call the two limiting values of x by the letter-names a and b, then the same would appear as

$$\mathrm{X}_b - \mathrm{X}_a = \int_a^b \mathrm{X}' dx\,.$$

Or, if the limiting values of x were, say, 15 and 85 feet, it would be written

$$\mathrm{X}_{85} - \mathrm{X}_{15} = \int_{15}^{85} \mathrm{X}' dx\,.$$

It must be noted that the limits which are written in always refer to values of the variable whose increment or differential appears in the integration. Thus the a and b or the 85 and 15 above mean invariably values of x, **not** values of X nor of X′.

46. Linear Graphic Diagrams of Integration. In figs. 5 and 6 are given two methods of graphically representing this process of integration. The first corresponds with the illustrations we have already employed. Here the curve xX is supposed to be built up step by step by drawing in each small stretch of horizontal length δx at a gradient equal to the known mean gradient X′ for that length. The gradient X′ is supposed known for each value of x, and its mean value throughout each very small length δx is therefore known.

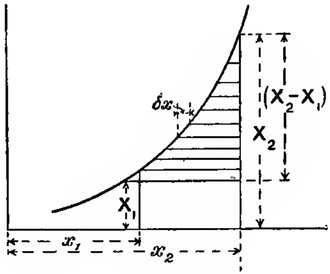

Fig. 5.

With regard to this statement it should be noted that a curve does not really possess a gradient **at a point**, but only **throughout a short length.** When we speak of the slope of a curve at a point, what we really mean is the slope of a minute portion

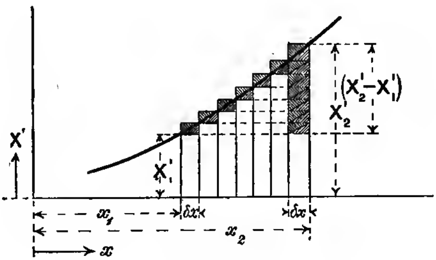

Fig. 6.

of its length lying partly in front and partly beyond the point: that is, there is actually no difference of meaning between the phrases "the slope of the curve at the point" and "the mean gradient throughout a short length at this point." Since each increment of X, or δX, equals X′ times the corresponding increment of x or δx, we have in fig. 5 all these increments of X

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There is, however, nothing meaningless or impossible in $\int dX$ at the same place. In fig. 1 up the vertical face under J, the δX's, or δh's, have the same concrete, finite meaning as they have elsewhere. Thus it is clearly improper to write $\int dX = \int X'dx$ for this part of the integration; the formula, which is *true in general*, fails under these special conditions.

51. Change of Form of Integral.—If L be a function of the variable l, and if its l-gradient be called L', then $\delta L = L'\delta l$; and if λ be any other function of l, then $\int \lambda dl = \int \frac{\lambda}{L'} dL$. When λ and L' are both capable of simple expression in terms of L, the latter form of the integral may be more easily dealt with than the former. Such a transformation of an integral is called a **change of the independent variable** or "**substitution.**" *

52. Definite and Indefinite Integrals.—Sometimes the limits of the integration are not expressed in the written symbol, which then stands simply $\int X'dx$. When thus written, it is understood that in the integration the variable x increases continuously up to an **undefined** limiting value, which is to be written x in the expanded form of the integral. In fact by $\int X'dx$ is meant $\int^x X'dx$, the upper limit being **any** final value of the gradually increasing x.

The lower limit may be written without defining the upper limit. Thus $\int_a X'dx$ means $(X - X_a)$. If various upper limiting values of x be successively taken, the part of the integral function involving a remains unchanged.

Such an integral may be written

$$\int X'dx = X + C,$$

that is, as the sum of two terms, one of which, C, remains unchanged when the upper limit is varied, while the other, X, remains the same although the lower limit be changed. This is called the **indefinite integral,** and C is called the **constant of integration.**

When both upper and lower limits are particularised, as in $\int_a^b X'dx$, the quantity is called a **definite integral.**

53. Integration Constant.—To show the exact meaning of the

* See Classified List, II. G.

integration constant C, compare the above two forms of writing the indefinite integral. The values of X being the same in both cases, it is clear that C equals $(-X_a)$. The integration constant, therefore, depends on the implied lower limit of x $(=a)$. If C be given, the implied lower limit a is thereby fixed; and conversely, if a be given, its value determines that also of C.

The indefiniteness of the indefinite integral may, therefore, be considered as due to free choice being left as to either or both limits. The part X depends on the choice of the upper limit, and remains indefinite so long as that is not fixed. The part C depends on the lower limit, and is indefinite until this limit is fixed.

54. Meaning of Integration Constant.—Figs. 7 and 8 may help to elucidate further this question of limits and of integration constant. In fig. 7 the same curve is drawn thrice in different positions in the diagram. P′Q′R′ is PQR simply raised at every point

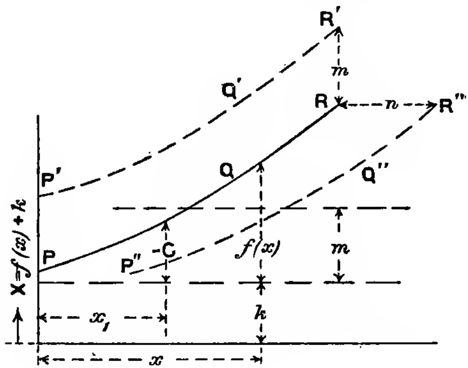

Fig. 7.

through the height m. P″Q″R″ is the same as PQR shifted horizontally the distance n. Since X′ is the gradient of the curve, the same values of the integral $\int X'dx$ will be obtained from all three curves if it be taken between limits on each which give the same series of values of X′. Thus the integrals will be the same when obtained from PQR and from P′Q′R′ if the same limits of x be used in each case. They will be the same from PQR and from P″Q″R″ if the same limits of X be used, which will mean limiting values of x in P″Q″R″ greater by n than those in PQR.

These upward and right-hand horizontal shiftings of the curve are equivalent to equal downward and left-hand horizontal shiftings of the axes from which the co-ordinates X and x are measured. Thus the two shiftings are combined in fig. 8. Here, in order to

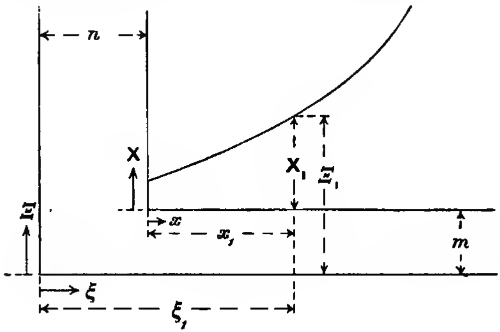

FIG. 8.

obtain the same series of values of X′, and thus the same value of the integral $\int X'dx$ when the co-ordinates are measured from one set of axes as when they are measured from the other, both upper and lower limiting values of x must be greater by n when measured from one set of axes than when measured from the other set; and this will mean also that the two limiting values of X are both greater by m for the one set of axes than for the other.

It is thus clear that the **limits** of a definite integral refer to defined points on the curve at which occur particular stated **gradients** (X′), that is, values of the **function to be integrated**; and *not* to points with particular stated values of either the variable (x) or of the integral (X). Since the function to be integrated (X′) is stated in terms of the variable (x), therefore the desired limits of X′ are conveniently named by reference to the corresponding values of x; but these latter really depend upon the arbitrarily chosen axes of reference.

55. Meaning of Integration Constant.—In the indefinite integral $\int X'dx = X + C$, if the part X is understood to mean the whole integral function at the undefined upper limit and measured from an arbitrarily chosen datum level, then X will include some constant such as m in figs. 7 and 8. But if X be understood to mean the sum of those terms of the integral function which involve x, excluding all terms in which x does not appear; then X will have a value independent of the arbitrary datum level from which X is measured (so that m will not appear): but, on the other hand, terms involving n will appear, *i.e.*, the value of X will depend on the position of the axis from which the x's are measured.

56. Extension of Meaning of Integration.—In the expression $X + C$, taking X to represent the whole integral quantity, it is often easy to select the axes of reference so as to give a specially convenient value to C.

For example, if the integral X be the deflection of a symmetrically designed girder, symmetrically loaded and supported at its two ends, the symmetry of the problem makes it *a priori* evident that equal deflections occur at equal distances on either side of the centre of the girder length; and, if this centre be chosen as the origin from which to measure the x's, equal deflections will be found for equal + and − values of x, and this condition, if applied, will be sufficient to determine one integration constant.

57. Integration the Inverse of Differentiation.—It is important to recognise that the operation of integrating a known function X' can lead to only one definite result, or rather to a series of results which differ from each other only in the value of the constant C.

If through any point of $P'Q'R'$ in fig. 7 there were drawn any curve deviating at any part from $P'Q'R'$, it is evident that, at some parts at least of this other curve, it would have a gradient different from that of the part of PQR lying immediately below or above it. Thus all curves having the same X' for each x are related to each other as are PQR and $P'Q'R'$. Now these two curves taken between the same limits give the same difference of height; and, if the integral be taken in the "indefinite" form, as meaning the total height at any x from the horizontal axis, the heights of the two curves differ everywhere by the same amount m, that is, the two indefinite integrals differ only in the value of the constant C.

58. Usual Method of finding New Integrals.—Since, then, a definite function of x when integrated with respect to x gives one and one only function of x (definite in all but the particular value of the additive constant), therefore, if we happen to know of a curve which gives at each point an x-gradient equal to the function of x which we wish to integrate, we conclude with certainty that the height of that curve is the integral sought for. The level of the axis from which the height is to be measured is fixed by the special, or "limiting," conditions of the problem.

This is the usual method of finding integrals: namely, we make use of our previous knowledge of differentials. The finding of differential coefficients, or gradients, is an easier process in general than the reverse or inverse process of finding integrals. This difference is analogous to that we observe in ordinary geometry, in which it is simpler to find the tangent of a known curve than it is to find the curve by building it up from a knowledge of the direction of its tangent at each point.

In some simple cases, examples of which will be given in the next chapter, it is as easy to find the integral directly as it is to find the differential. Very often the study of the character of a curve or law leads to the simultaneous recognition of the differential and the inverse integral.*

CHAPTER III.

EASY AND FAMILIAR EXAMPLES OF INTEGRATION AND DIFFERENTIATION.

59. Circular Sector.—An easy example of integration is the finding the area of a circle, or of any sectorial part of a circle.

In fig. 9 there is drawn an isosceles triangle of very small vertical angle placed at O, the centre of a circle, and whose correspondingly small base is a very short tangent to the arc of the circle. The height of this triangle is r, the radius of the circle. Its base is, in a minute degree, longer than the arc, but equals the arc in length with an approximation which is closer as the vertical angle at O becomes smaller. Taking δp, the peripheral arc length, as an approximation to the base, the area is $\frac{1}{2}r.\delta p$. The whole angle AOB may be split up into a very large number of minute angles, in each of which may be formed a small triangle

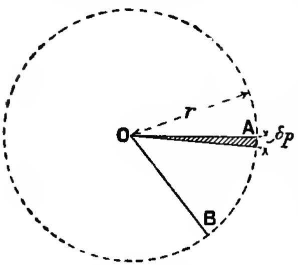

FIG. 9.

similar to the above. The sum of the areas of these triangles is greater than that of one of them in the same ratio that the sum of their bases is greater than the base of one of them, because the factor $\frac{1}{2}r$ has the same value in all. The series of bases form a connected chain of very short tangents lying outside the arc AB. As the individual links of this chain become shorter, and the total number of them correspondingly greater, the sum of their lengths becomes equal to the arc length AB with closer and closer approximation, and, at the same time, the sum of the triangular areas

* See Appendix A.

equals that of the circular sector ABO with closer and closer approximation. Thus, taking the arcs minutely short, and calling the arc length AB by the letter p, we find

$$\text{Circular sectorial area} = \tfrac{1}{2}r.p\,.$$

For the complete circular area the length of arc p becomes the complete circumference of the circle or $2\pi r$. Thus the complete circular area is πr^2.

Here the important point to note is, that the approximation of the sum of the triangular areas to equality with the circular area proceeds *pari passu* with that of the sum of the tangent lengths to the circular arc length.

60. Constant Gradient.—In this integration the factor $\frac{1}{2}r$ is a constant. The variable is p, the arc length measured from some given point on the circular circumference. The increment of p is δp: that is, δp may be looked upon as the increase of the arc length p, as the sectorial area is swept out by a radius revolving round the centre O. The integral is the gradually increasing area so swept out. The increment, or differential of this area, is $\frac{1}{2}r.\delta p$. Thus $\frac{1}{2}r$ is the differential coefficient of the area with respect to p; or $\frac{1}{2}r$ is the p-gradient of the area. Calling the constant $\frac{1}{2}r$ by the letter k, we may write this

$$\frac{d}{dp}(kp) = k\,;\ \text{or otherwise}\ \int kdp = kp\,.^*$$

61. Area of Expanding Circle.—A second simple example of integration is that of the area swept through by the circumference of a gradually expanding circle. Let the radius from which the expansion begins be called r_1: the area inside this initial circumference is πr_1^2. At any stage of the expansion when the radius has become any size r, the area swept through has been $(\pi r^2 - \pi r_1^2)$. This is the indefinite integral; the constant of integration being here $-\pi r_1^2$.

As r increases by δr from $(r - \frac{1}{2}\delta r)$ to $(r + \frac{1}{2}\delta r)$, the area swept out by the circumference is a narrow annular strip whose mean peripheral length is $2\pi r$, and whose radial width is δr, as shown on fig. 10. The area of this annular strip, therefore, equals $2\pi r.\delta r$, and this is the increment, or differential, of the integral area swept out. The "differential coefficient with respect to r," or the "r-gradient," of the area is, therefore, $2\pi r$. That is, the r-gradient of $(\pi r^2 + \mathrm{C})$, where C is any constant, is $2\pi r$. Otherwise written,

$$\frac{d}{dr}(\pi r^2 + \mathrm{C}) = 2\pi r\,.$$

* See Classified List, I. 4, and III. A. 1.

Here π is a constant multiplier, and, if any other constant multiplier k were used, the result would be

$$\frac{d}{dr}(kr^2 + C) = 2kr.$$

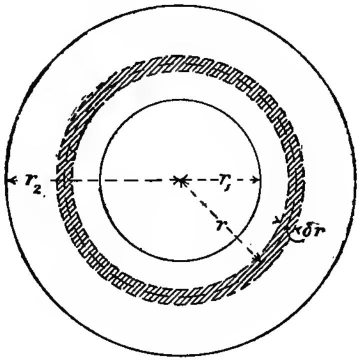

FIG. 10.

Written in the inverse manner, the same result appears as

$$\int 2krdr = kr^2 + C.*$$

62. Rectangular Area.—A third simple example is shown in fig. 11. Here a vertical line of length k is moved horizontally to the right from an initial position x_1. Its lower end moves along the axis of x. Its upper end moves along a horizontal straight

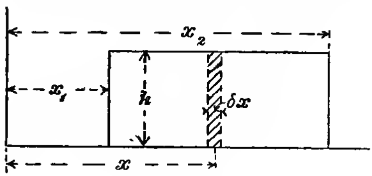

FIG. 11.

line at the height k above this axis. The vertical sweeps out a rectangular area equal to k multiplied by the length of horizontal movement. At any stage x of the movement the area swept out

* See Classified List, III. A. 2.

is $k(x-x_1)$. Here the variable is x. Call its increment δx. While the vertical moves horizontally δx, the area swept out is a narrow rectangular strip equal to $k\delta x$. This is the increment of the integral area. Thus the x-gradient of the area is k. Calling the constant $-kx_1$ by the letter C, this result may be written in the two forms

$$\frac{d}{dx}(kx+\text{C})=k$$

and

$$\int kdx = kx + \text{C}\,.\text{*}$$

This result is identical in form with that of § 60, fig. 9, except that in the latter the integration constant is zero and does not appear, and the variable is named p, while x is the name adopted in fig. 11. The choosing of one or other name to indicate the variable is, of course, of no consequence; but taken as two illustrations of purely geometric integration, the two results are of *different kinds*, although taken as formulas simply they are identical.

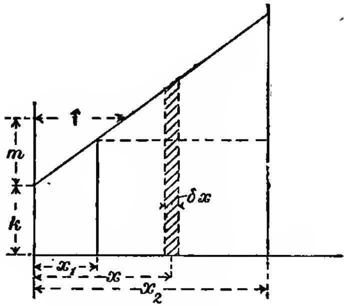

FIG. 12.

63. Triangular Area.—In fig. 12 we have an area swept out by a vertical line as it moves from left to right, and increases in height at a uniform rate during the motion. Its lower end lies always in the axis of x. Its upper end lies in a straight line whose inclination to the x-axis is called m.

During the movement δx, from $(x-\frac{1}{2}\delta x)$ to $(x+\frac{1}{2}\delta x)$, the mean height of the small strip of area swept out is $(k+mx)$, and its area, therefore, is

$$(k+mx)\delta x\,.$$

The whole area swept out during the motion from the lower limit x_1 to any upper limit x may be divided into two parts; one rectangular with the base $(x-x_1)$ and the height $(k+mx_1)$; the other triangular with the same base and the height $m(x-x_1)$.

* See Classified List, III. A. 1.

This whole area is, therefore,

$$(x - x_1)\{(k + mx_1) + \tfrac{1}{2}m(x - x_1)\} = k(x - x_1) + \tfrac{1}{2}m(x^2 - x_1^2).$$

This may be written in the form

$$\Big[kx + \tfrac{1}{2}mx^2\Big]_{x_1}^{x}$$

which form means that the function of x standing inside the square bracket is to be calculated for the two values of x, x and x_1 (the former "indefinite" or not particularised, the latter special), and the latter value of the function subtracted from the former.*

Again, it may be written in the other form

$$kx + \tfrac{1}{2}mx^2 + C$$

where C is a "constant of integration."

This result may be expressed by either of the two following equations:—

$$\frac{d}{dx}\left\{ kx + \tfrac{1}{2}mx^2 + C \right\} = k + mx$$

and

$$\int (k + mx)dx = kx + \tfrac{1}{2}mx^2 + C.$$

64. First and Second Powers of Variable.—The last integral may be split into two parts. The first is

$$\int kdx = kx + C_1$$

which is identical with what is obtained in fig. 11. The second is

$$\int mxdx = \tfrac{1}{2}mx^2 + C_2$$

which is the sweeping out of the triangular area.

65. Integral Momentum.—The following are other easy examples of the first of these two formulas.

The extra momentum acquired by a mass m in the interval between time t_1 and time t_2, during which its velocity is accelerated at the constant rate g, if its velocity be v_1 at time t_1, is

$$m(v_2 - v_1) = \int_{t_1}^{t_2} mgdt = mg(t_2 - t_1).$$

* See Classified List, "Notation."

Here mg is the acceleration of momentum, or the time-gradient of the momentum.

66. Integral Kinetic Energy.—The simultaneous increase of Kinetic Energy is

$$\frac{m}{2}(v_2^2 - v_1^2) = \frac{m}{2}\{g^2(t_2 - t_1)^2 + 2g(t_2 - t_1)v_1\}$$

$$= mg(t_2 - t_1)\{v_1 + \tfrac{1}{2}g(t_2 - t_1)\}$$

= Extra Acquired momentum × Average Velocity during interval.

67. Motion integrated for Velocity and Time.—Again the distance travelled by a train between the times t_1 and t_2, when running at a constant velocity v, is

$$\int_{t_1}^{t_2} v dt = v(t_2 - t_1).$$

Here the velocity v is the time-gradient of the distance travelled.

68. Motion from Acceleration and Time.—Easy examples of the second formula are the following:—

If the velocity of a mass be accelerated at the uniform rate g; then, since the velocity at any time t is $\{g(t - t_1) + v_1\}$, and since in a small interval of time δt, the distance travelled is $v.\delta t$, where v is the average velocity during δt, we find the distance travelled in interval $(t_2 - t_1)$ to be

$$\int_{t_1}^{t_2} \{gt - gt_1 + v_1\}dt = \frac{g}{2}(t_2^2 - t_1^2) - gt_1(t_2 - t_1) + v_1(t_2 - t_1)$$

$$= (t_2 - t_1)\{v_1 + \tfrac{1}{2}g(t_2 - t_1)\}.$$

If this be multiplied by mg, we get again the increase of kinetic energy as shown above in § 66; so that the increase of kinetic energy equals the uniform acceleration of momentum (mg) multiplied by the distance travelled.*

69. Bending Moments.—As another example, take a horizontal beam loaded uniformly with a load w per foot length. If we name by the letter l lengths along the beam from any section where we wish to find the bending moment due to this load; then on any short length δl there is a load $w.\delta l$, and the moment of this load upon the given section is $wl.\delta l$, where l means the length to the

* See Appendix B.

middle of δl. The integral, or total, moment exerted upon this section by the part of the load lying between l_1 and l_2 is

$$\int_{l_1}^{l_2} wldl = \frac{w}{2}(l_2^2 - l_1^2)$$

$$= w(l_2 - l_1) \times \tfrac{1}{2}(l_2 + l_1)$$

= whole load on $(l_2 - l_1)$ multiplied by the distance of the middle of the same length from the given section.

It must be noted that this is the moment exerted by the load alone independently of that exerted by the forces supporting the beam.

70. Volume of Sphere.—Passing now to volumetric integrals, we may consider a very small sectorial part of the volume of a sphere as an equal-sided cone of very small vertical angle placed at the centre of the sphere, and with a very small spherical base nearly coinciding with the flat surface of small area touching the sphere. The volume of the small cone with the flat base is known to be $\frac{1}{3}$ the product of its base area by its height. The height here is r, the radius of the sphere. This is true whatever be the shape of the cross section of the cone. Now the whole volume of the sphere is made up of a very large number of such small-angled cones **with spherical bases,** these cones fitting close together so as to fill up the whole space. They would not fit close together if their cross sections were, say, circular; but the argument does not depend on the shape of the cross section, and this is to be taken such as will make the cones fit close together. In all these small conic volumes, the common factor $\frac{1}{3}r$ appears as a constant: each is $\frac{1}{3}r.\delta A$, if δA represent the area of the small base. Thus the sum of the volumes is greater than any one of them in the same ratio as the sum of the areas of the bases is greater than the base-area of that one. Thus if A be the sum of the bases, or $\int dA = A$, we have the sum of the volumes equal to $\frac{1}{3}rA$. For any sectorial portion of the volume of the sphere, the sum of the areas of the flat tangent bases approximates to the area of the corresponding portion of the spherical surface *pari passu* with the approximation of the sum of the flat-based conical volumes to the sum of the round-based conical volumes, which latter is the true spherical volume. Thus, if A be the area of the spherical surface, the volume subtended by it at the centre is $\frac{1}{3}rA$. If A be taken as the complete spherical surface, then $\frac{1}{3}rA$ is the total spherical volume. This integration is in form identical with that of fig. 9. It differs from that in **kind**, inasmuch as the differential δA is an area, while in fig. 9 the differential δp is a line. The mathematical process is the same in both cases; but the legitimacy of the application of this process depends in the one case upon the physical relations between certain curved and straight lines, while in the other case it depends on the physical relations between certain curved and flat surfaces.

When it is known that the ratio of the surface-area of a sphere to the square of its radius is 4π, the above integration proves the complete spherical volume to be $\frac{4}{3}\pi r^3$ (see § 76 below).

71. Volume of Expanding Sphere.—Consider now the spherical volume as swept through by the surface of a gradually expanding sphere. If the radius be r_1 at one stage of the expansion, and r at another, the volume swept through between these two stages is $\frac{4}{3}\pi(r^3 - r_1{}^3)$. During any small increase of size δr from the radius $(r - \frac{1}{2}\delta r)$ to $(r + \frac{1}{2}\delta r)$, the volume swept out is the normal distance δr between the smaller and larger spherical surfaces multiplied by the mean area of the spherical surface during the motion, viz., $4\pi r^2$. That is, the increment of volume is

$$4\pi r^2 \delta r.$$

The **definite** integral of this is, as above stated,

$$\left[\tfrac{4}{3}\pi r^3\right]_{r_1}^{r}$$

and the **indefinite** integral for an indefinite size r is

$$\tfrac{4}{3}\pi r^3 + C.$$

Thus $4\pi r^2$ is the r-gradient of $(\frac{4}{3}\pi r^3 + C)$.

If x were used to represent the radius, and X the volume, and X′ the x-gradient of X and the constant factor 4π be written k: we would here have

$$X' = 4\pi x^2 = kx^2$$

$$X = \int kx^2 dx = \frac{k}{3}x^3 + C.*$$

Expressed in words, the radius-gradient of the spherical volume is the spherical surface.

72. Volume of Expanding Pyramid.—Consider a rectangular-based pyramid of height x, and the two sides of whose base are mx and nx. The area of the base is mnx^2, and, therefore, the pyramidal volume is $\frac{1}{3}mnx^3$. Now, suppose the size of this pyramid to be gradually increased, keeping its shape unaltered, by extending

* See Classified List, III. A. 2.

its sides in the same planes, and moving the base away from the vertex while keeping the base always parallel to its original position. As the height x increases, the sides of the rectangular base both increase in the same ratio so as to remain always mx and nx; and, therefore, the increasing volume is always equal to $\frac{1}{3}mnx^3$. As the base moves a distance δx away from the vertex from the height $(x-\frac{1}{2}\delta x)$ to $(x+\frac{1}{2}\delta x)$, the increase of volume thus added to the pyramid is the mean area of the base during this motion, viz., mnx^2, multiplied by the normal distance δx between the old and the new bases. The increment of volume is thus $mnx^2.\delta x$. The definite integral volume taken between the limit x_1 and x_2 is

$$\left[\tfrac{1}{3}mnx^3\right]_{x_1}^{x_2}.$$

If the constant factor mn be written k, this result would be thus expressed, taking the indefinite form of the integral :—

$$\int kx^2dx = \tfrac{1}{3}kx^3 + \text{C},$$

which is formally or symbolically identical with the last result obtained. The difference between the two in kind is perhaps best recognised by comparing the word-expression of the last result with the following similar statement of our present one :—

The height-gradient of the volume of a pyramid of given shape is the area of the base of the pyramid.

In this last statement of the result no reference is made to the special shape of the cross section of the pyramid, and it is readily perceived that the reasoning employed above did not depend in any degree upon the rectangularity of the base.

73. Stress Bending Moment on Beam.—Take as another example of this formula leading from the second power in the gradient to the third power in the integral, the calculation of the stress-bending moment of a rectangular beam section exposed to pure bending of such degree as produces only stresses within the elastic limit. Under this condition the normal stress on the section increases uniformly with the distance from the neutral axis, which in this case is at the middle of the depth. Thus, if the whole depth of the section be called H, and the intensity of stress at the top edge (at distance $\frac{\text{H}}{2}$ from neutral axis) be called k; then the intensity of stress at any distance h from the axis is $k\frac{h}{\frac{1}{2}\text{H}} = \frac{2k}{\text{H}}h$.

If the width of the section be B, the area of a small cross strip of it, of depth δh, is $B\delta h$. If h mean the height to the middle of δh, then the whole normal stress on this strip is $\frac{2kB}{H} \cdot h\delta h$, and the moment of this round the neutral axis is $\frac{2kB}{H} h^2 . \delta h$, because h is the leverage. The sum of these moments over the half of the section lying above the axis is the integral of this between the limits $h = 0$ and $h = \frac{1}{2}H$, or

$$\int_0^{\frac{1}{2}H} \frac{2kB}{H} h^2 dh = \left[\frac{2kB}{3H} h^3\right]_0^{\frac{H}{2}}$$
$$= \frac{2kB}{3H} \cdot \frac{H^3}{8}$$
$$= \frac{k}{12} BH^2 .$$

An equal sum of moments of like sign is exerted by the stresses on the lower half of the section, and thus the

Total Stress Bending Moment $= \frac{1}{6}kBH^2$.*

74. Angle Gradients of Sine and Cosine and Integration of Sine and Cosine.—In fig. 13 the angle α is supposed measured in radians, that is, in circular measure, the unit of which is the angle whose arc equals the radius. Radians, sines, cosines, tangents, etc., are pure numbers, or ratios between certain lengths and the radius of a circle; but if the radius be taken as unity, as in fig. 13, then these ratios are properly represented by lengths of lines, this graphic representation being to an artificial scale just as, to other artificial scales, velocities, moments, weights, etc., can be graphically represented by line-lengths. In fig. 13 the angle α is measured to such a scale by the length of the arc Na, while to the same scale $\sin \alpha$ is measured by as and $\cos \alpha$ by ac. Take a very small angle $\delta\alpha$, and mark off from N the two angles $(\alpha - \frac{1}{2}\delta\alpha)$ and $(\alpha + \frac{1}{2}\delta\alpha)$. The horizontal and vertical projections of $\delta\alpha$ (parallel to as and ac) are evidently the increments of the sine and cosine

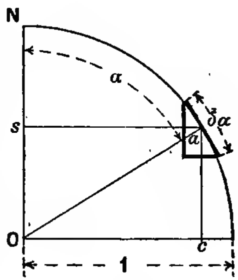

FIG. 13.

* See Appendix C.

for the angle increment δa. The horizontal projection is a positive increment of the sine; the cosine decreases as a increases, so that the vertical projection is the decrement or negative increment of the cosine. If δa be taken small enough to justify the short arc being taken as a straight line, δa and its two projections form a small right-angled triangle of the same shape as **Oas**. We have, therefore,

Increment of $\sin a = \delta(\sin a) =$ Horizontal projection of δa

$$= \frac{ac}{a0}.\delta a = \cos a \, \delta a$$

and

Decrement of $\cos a = -\delta(\cos a) =$ Vertical protection of δa

$$= \frac{as}{a0}.\delta a = \sin a \, \delta a \, .$$

Integrating these increments between any limits a_1 and a_2, the results are

$$\int_{a_1}^{a_2} \cos a \, da = \sin a_2 - \sin a_1$$

$$\text{and} \int_{a_1}^{a_2} \sin a \, da = \Big[-\cos a \Big]_{a_1}^{a_2} = \cos a_1 - \cos a_2 \, .$$

The student should carefully follow out this integration on the diagram through all four quadrants of the complete circle, paying attention to the changes of sign.

Written as indefinite integrals these results are

$$\int \cos a \, da = \sin a + C$$

$$\text{and} \int \sin a \, da = C - \cos a \, .\text{*}$$

Expressed in words, this is, the angle-gradient of the sine of an angle is its cosine, and that of its cosine is its sine taken negatively.

75. Integration through 90°.—Since $\sin 0° = 0$ and $\cos 0° = 1$, while $\sin 90° = 1$ and $\cos 90° = 0$, we find, integrating between the limits 0° and 90°,

$$\int_{0°}^{90°} \cos a \, da = 1$$

$$\text{and also} \int_{0°}^{90°} \sin a \, da = 1 \, .$$

* See Classified List, VI. 1 and 2.

76. Spherical Surface.—Let this result be applied to the calculation of the area of the earth's surface, assuming it to be spherical. The whole surface may be divided up into narrow rings of uniform width lying between parallels of latitude. Thus, if the difference of latitude be taken to be $\frac{1}{4}°$, the uniform width of each ring will be about $17\frac{1}{4}$ miles. The meridian arc throughout this length may be considered straight without appreciable error. The ring at the equator forms practically a cylindrical ring of radius equal to that of the earth, R. A ring taken at latitude λ has a mean radius $R \cos \lambda$; and the circumferential length of its centre line is therefore $2\pi R \cos \lambda$. Naming the difference of latitude for one ring $\delta\lambda$, the width of the ring is $R.\delta\lambda$, and its area therefore $2\pi R \cos \lambda . R . \delta\lambda = 2\pi R^2 \cos \lambda . \delta\lambda$. The factor $2\pi R^2$ being the same for all the rings, we may first sum up all the products $\cos \lambda . \delta\lambda$, and afterwards multiply this sum by the common factor $2\pi R^2$. If we perform this integration from the equator to the north pole, that is, between the limits $\lambda = 0°$ and $\lambda = 90°$, we obtain the surface of the hemisphere. The integral of $\cos \lambda . \delta\lambda$ from $0°$ to $90°$ is 1; and therefore the hemispherical surface is $2\pi R^2$, and the whole spherical surface $4\pi R^2$. We used this result in § 70, p. 35.

77. Spherical Surface integrated otherwise.—The above total is $2\pi R \times 2R$. Here $2\pi R$ is the circumference of a cylinder touching the sphere, and $2R$ is the diameter of the sphere; so that the whole spherical surface equals that of a touching cylindrical surface whose length equals the diameter (or length) of the sphere.

In fig. 14 this circumscribing cylinder is represented by its axial section *nn*, *ss*. For each strip of spherical surface of radius r bounded by parallels of latitude $\lambda\lambda$, $\lambda\lambda$, there corresponds a strip of cylindric surface *ll*, *ll* of radius R, which latter is, in fact, the radial projection on the cylindric surface of the spherical strip. It is easy to prove that the arc $\lambda\lambda$ is greater than its projection *ll* in the same ratio that R is greater than r. Hence the areas of the two differential strips are equal; and, therefore, the integral areas from end to end are also equal. This proof is more elementary than that given in the previous paragraph.

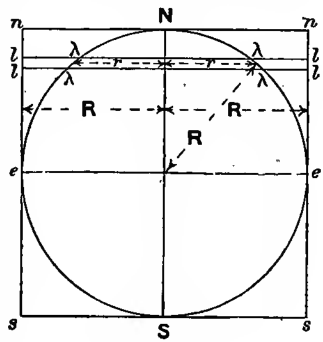

FIG. 14.

78. Angle-Gradients of Tangent and Co-tangent and Integration of Squares of Sine and Cosine.—In fig. 15, a small angle-increment δa is marked off equally below and above the angle a, and radii are drawn from centre O through the extremities of δa out to meet the two tangents to the quadrant of the circle at N and E. The tangent of a, or tan a, is measured along the tangent

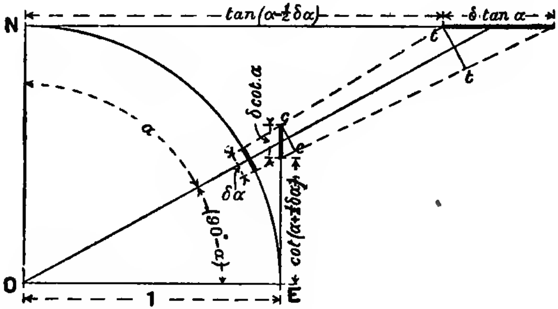

FIG. 15.

from N to the radius at a, and its co-tangent, or cot a, along the tangent at E to the same radius. The increments of tan a, and of cot a, due to δa, are marked on the figure. δ tan a is a positive increase of tan a for a positive increase of the angle, while δ cot a is a decrease of cot a. The lines *tt* and *cc* are drawn parallel to the short arc δa. *tt* is therefore inclined to $\delta \tan a$ at the angle a, and *cc* to $\delta \cot a$ at the angle $(90° - a)$. Therefore

$$tt = \cos a . \delta \tan a \quad \text{and} \quad cc = -\sin a . \delta \cot a .$$

Here the $-$ sign is used in order to make *cc* positive ($\delta \cot a$ being negative).

Now *tt* is greater than δa in the ratio of O*t* to the radius of the circle, or $\frac{Ot}{ON}$. Similarly *cc* is greater than δa in the ratio $\frac{Oc}{OE}$. That is,

$$\delta a = tt . \cos a = cc . \cos (90° - a) = cc \sin a .$$

Therefore,

$$\delta a = \cos^2 a . \delta \tan a = -\sin^2 a . \delta \cot a .$$

Taking all the increments minutely small, these results are written

$$\frac{d \tan \alpha}{d\alpha} = \text{the angle-gradient of the tangent}$$
$$= \frac{1}{\cos^2 \alpha}$$

and

$$\frac{d \cot \alpha}{d\alpha} = \text{the angle-gradient of the co-tangent}$$
$$= -\frac{1}{\sin^2 \alpha}.$$

Or otherwise

$$\int \frac{1}{\cos^2 \alpha} . d\alpha = \tan \alpha + C$$

and

$$\int \frac{1}{\sin^2 \alpha} d\alpha = C - \cot \alpha .^*$$

79. Gradient of Curve of Reciprocals.—In figure 16 there is

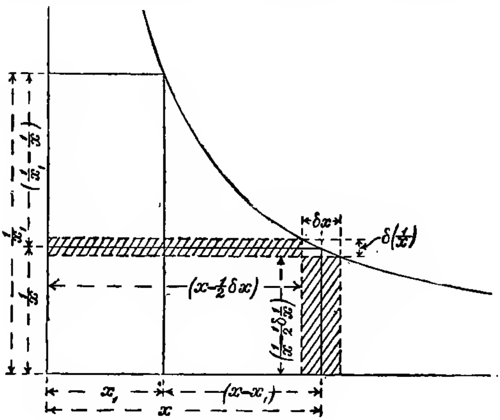

Fig. 16.

drawn a curve of reciprocals; the horizontal ordinate being x, the vertical ordinate is $\frac{1}{x}$.

* See Classified List, VI. 11 and 12.

The area of the rectangle formed by the axes and the ordinates at any point is $x \times \frac{1}{x} = 1$; constant for all points of the curve. These two rectangles at the two points x_1 and x overlap each other, having the common area $x_1 \times \frac{1}{x}$ as part of each. Subtract this common part and there is left

$$x_1\left(\frac{1}{x_1} - \frac{1}{x}\right) = (x - x_1)\frac{1}{x}$$

or

$$\frac{\frac{1}{x_1} - \frac{1}{x}}{x - x_1} = \frac{1}{x_1 x}.$$

This is the ratio of the *decrease* of $\frac{1}{x}$ to the *increase* of x. When the increments are made minutely small, $\frac{1}{x_1 x}$ becomes practically $\frac{1}{x^2}$. In the figure a small increment of x, viz., δx, is set off equally below and above x. The above equality of areas means the equality of the two narrow strips of area rafined over in the figure. The equality is, therefore,

$$\{x - \tfrac{1}{2}\delta x\}.\delta\left(\frac{1}{x}\right) = \left\{\frac{1}{x} - \tfrac{1}{2}\delta\left(\frac{1}{x}\right)\right\}\delta x.$$

Adding $\frac{1}{2}\delta x.\delta\left(\frac{1}{x}\right)$ to each side and writing $\frac{d\frac{1}{x}}{dx}$ instead of $\frac{\delta\left(\frac{1}{x}\right)}{\delta x}$; changing also the sign, because $\frac{1}{x}$ decreases while x increases, we have

$$\frac{d\frac{1}{x}}{dx} = -\frac{1}{x^2}.$$

Expressed in words this is:—The x-gradient of the reciprocal of x is minus the reciprocal of the square of x. Writing this result inversely, we have

$$\int \frac{dx}{x^2} = C - \frac{1}{x} \, *$$

* See Classified List, III. A. 2.

where C is the integration constant to be determined by special limiting conditions.

80. x-Gradient of Xx and Inverse Integration. Formula of Reduction.—In fig. 17 there is drawn a curve whose ordinates are called x and X. X represents any function of x. x is taken to the middle of δx; and, since the arc-length corresponding to δx is of minute length and may therefore be considered as straight, the point xX on the curve bisects this arc-length and also bisects δX. Also the horizontal and vertical lines through the point xX on the curve divide the rectangular area ab into four equal parts, each $\frac{1}{4}\delta x.\delta$X.

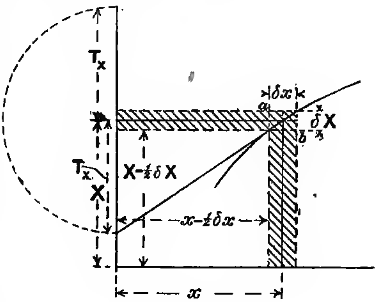

Fig. 17.

The increase of the rectangular area Xx due to the increase δx of x is, therefore,

$$(X+\tfrac{1}{2}\delta X)(x+\tfrac{1}{2}\delta x) - (X-\tfrac{1}{2}\delta X)(x-\tfrac{1}{2}\delta x) = X\delta x + x\delta X$$

by actual multiplication, the first and fourth terms of each product cancelling out.

The first of these two terms of this increment is the strip of area between the two dotted verticals of height X; the second is the strip between the two dotted horizontals of length x. These two strips overlap each other by the $\frac{1}{4}(ab)$ small rectangle, and this has to be taken twice to obtain their sum. This compensates for the two strips not covering the outer small $\frac{1}{4}(ab)$ rectangle.

Dividing by δx, and taking minutely small increments, that of (Xx) being called d(Xx), and the x-gradient of X being called X′, there results

$$\frac{d(Xx)}{dx} = X + X'x.$$

According to §§ 38 and 40, pp. 16 and 17, the X-subtangent measures X′x; therefore the present x-gradient equals the sum of the function X and its X-subtangent. This X-subtangent is shown in fig. 17, where it is also graphically added to X.

The result written in the inverse integration-symbolism is

$$\int (X + X'x)dx = Xx + C.$$

As explained below in § 83, the integral $\int(X + X'x)dx = \int X dx + \int X'x dx$. Therefore the result of this article may be written

$$\int X dx = Xx - \int X'x dx + C.$$

This is an important "Reduction Formula." *

81. x-Gradient of X/x and Inverse Integration.—In fig. 18 a curve is drawn whose ordinates are called x and X, any function of x. From two points x,X and $(x + \delta x)$, $(X + \delta X)$

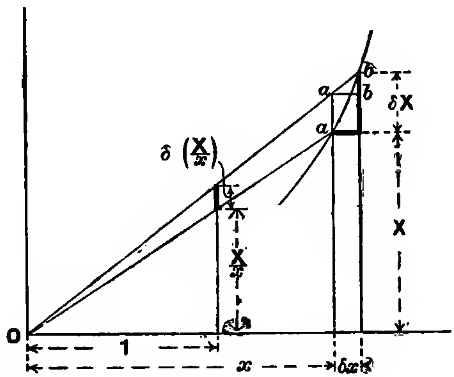

Fig. 18.

on this curve are drawn two straight lines to the origin O; and on these two lines lie the upper extremities of verticals drawn at the horizontal distance 1 from O. Evidently these last verticals measure the ratios $\frac{X}{x}$ and $\frac{X + \delta X}{x + \delta x}$. The difference between them is the increase of the ratio $\frac{X}{x}$ due to the increase δx of x, and is marked $\delta\left(\frac{X}{x}\right)$ in the figure. It is less than the small height aa in the ratio of 1 to x; and this height aa is less than δX by bb. This small height bb bears the same ratio to δx as $(X + \delta X)$ bears to $(x + \delta x)$. Thus

* See Classified List, I. 8 and II. K.

$$\delta\left(\frac{X}{x}\right)=\frac{1}{x}\left(\delta X-\frac{X+\delta X}{x+\delta x}.\delta x\right).$$

Therefore, dividing by δx,

$$\frac{d\left(\frac{X}{x}\right)}{dx}=\frac{X'}{x}-\frac{X}{x^2}$$

where, since extremely minute increments are taken, $\frac{X}{x}$ is substituted for $\frac{X+\delta X}{x+\delta x}$.*

CHAPTER IV.

IMPORTANT GENERAL LAWS.

82. **Commutative Law.**—If $kX'\delta x$ is to be integrated, where to each $X'\delta x$ the same constant multiplier k is to be applied, it is evidently allowable to sum up first the series of products $X'\delta x$, and then to multiply this sum by k. Symbolically written this is

$$\int kX'dx=k\int X'dx$$

taken between the same limits in either case.†

Reverting to the graphic representation of integration in fig. 5, the proposition means that if there be two curves drawn, of which one has at each x its height k times the other, then the first has at each x its gradient also k times as steep as that of the other.

83. **Distributive Law.**—If there be two curves such as in fig. 6, the height of one being called X' and that of the other Ξ', then a third curve may be drawn, of which the height is $(X'+\Xi')$. The area under the first curve is $\int X'dx$; that under the second is $\int \Xi'dx$; that under the third is $\int(X'+\Xi')dx$. For each δx at the same x, the area of the narrow strip $(X'+\Xi')\delta x$ for the third curve equals the sum of the two strips $X'\delta x$ and $\Xi'\delta x$ for the first two

* See Appendix D. † See Classified List, I. 4.

curves. Since this is true of each strip, it is true of the whole areas; or,

$$\int X'dx + \int \Xi' dx = \int (X' + \Xi')dx\ *$$

the limits of x being taken the same in all three curves. The two curves may represent entirely different functions of x, subject only to the one condition that they must be of the **same kind**, it being impossible to add together quantities of different kinds. In both integration and differentiation this proposition is more frequently used by way of splitting up a whole integral into parts easier to deal with taken separately than by the converse process of combining parts into one whole. It may be extended to the more general formula

$$\int (X' + \Xi' + \mathfrak{X}' + \text{etc.})dx = \int X'dx + \int \Xi'dx + \int \mathfrak{X}'dx + \text{etc.}$$

If the separate integrals on the right-hand side of the last equation be called X, Ξ, $\mathfrak{X}$, etc.; then the differential view of the same proposition is that the

$$x\text{-gradient of } \{X + \Xi + \mathfrak{X} + \text{etc.}\}$$

$$= \frac{d}{dx}\left\{ X + \Xi + \mathfrak{X} + \text{etc.} \right\} = X' + \Xi' + \mathfrak{X}' + \text{etc.}$$

$$= x\text{-grad.}X + x\text{-grad.}\Xi + x\text{-grad.}\mathfrak{X} + \text{etc.}$$

Evidently the proposition of § 82 is only a special case of this in which $X' = \Xi' = \mathfrak{X}'$, etc., etc.

84. Function of a Function.—In fig. 19 there is drawn a curve, the horizontal and vertical ordinates of which are called l and L. Thus L is a function of l, the nature of the function being graphically described by this curve. A second curve is drawn, the vertical ordinates to which are the same L's as for the first curve (plotted and measured to the same scale), and whose horizontal ordinates are called λ. λ is a function of L, the curve graphically characterising the form of the function. λ is a quantity which may possibly be of the same kind as l, and, if so, it might be plotted to the same scale. But the general case is that in which λ is not of the same kind as l, and cannot possibly, therefore, be plotted to the same scale, although in the diagram it is measured in the same direction.

Since for each given value of L the second curve gives a definite

* See Classified List, I. 5.

corresponding value of λ, and the first curve gives a definite value of l; it follows that for each value of l there is a definite value of λ. In general there may be more than one value of λ for each l; but all the values of λ corresponding to one given value of l are definite. Thus λ is a definite function of l.

In the figure the l-gradient of L is represented graphically by a height obtained by drawing a tangent at the point lL, and plotting horizontally from this point a distance representing to the proper scale unity. This gradient is called L′ in the figure.

The L-gradient of λ is similarly represented, the unit employed being measured vertically, and not being the same as that used in

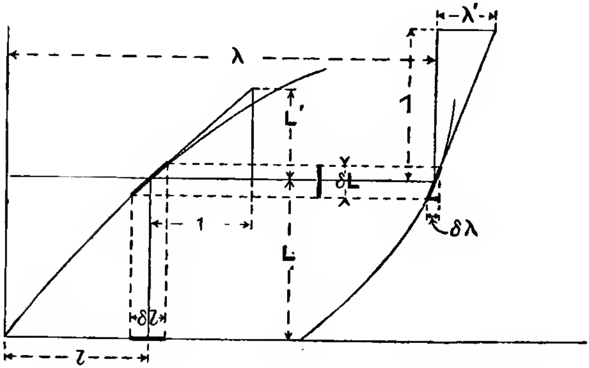

FIG. 19.

finding L′, because the scales involved are different. It is marked λ′. The two gradients shown in the figure are for the same value of L; that is, the tangents are drawn at points at the same level in the two curves.

If δL and δl are the two projections of any very short length of the first curve lying partly on each side of the point where the tangent is drawn, $L'=\frac{\delta L}{\delta l}$. If δL and δλ are the projections of any very short length of the second curve lying partly on each side of the point where the tangent is drawn, then $\lambda'=\frac{\delta\lambda}{\delta L}$.

If the two short arcs on the two curves be taken so as to give the same vertical projection, that is, the same δL, as is shown in fig. 19 by the dotted lines; then in the product L′λ′ the δL cancels out.

Thus,

$$\lambda' L' = \frac{\delta\lambda}{\delta L} \cdot \frac{\delta L}{\delta l} = \frac{\delta\lambda}{\delta l}$$

$$= \frac{d\lambda}{dl} \text{ when the increments are}$$

taken minutely small. In words this is :—The l-gradient of λ equals the L-gradient of λ multiplied by the l-gradient of L.

If we use the notation x, X, and F(X) instead of l, L, and λ; and if by F′(X) we understand the X-gradient of F(X) or $\frac{dF(X)}{dX}$; the same is written

$$\frac{dF(X)}{dx} = F'(X) \cdot X' \text{ or } \frac{dF(X)}{dX} \cdot \frac{dX}{dx}.*$$

85. Powers of the Variable; Powers of Sin and Cos.—This general proposition is one of the most fruitful of all laws in producing useful results when applied to particular functions, as will be seen in the next chapter.

Simple illustrations of its meaning are the following :—

Let

$$L = l^2 \text{ and } \lambda = L^3 \quad \therefore \lambda = l^6.$$

From §§ 64 and 71 we know that

$$L' = 2l \text{ and } \lambda' = 3L^2 = 3l^4$$

$$\therefore \frac{d\lambda}{dl} = \frac{dl^6}{dl} = 6l^5.$$

Written inversely,

$$\int l^5 dl = \frac{l^6}{6} \cdot \dagger$$

Again, let

$$L = l^3 \text{ and } \lambda = \frac{1}{L} \quad \therefore \lambda = \frac{1}{l^3}.$$

From §§ 71 and 79,

$$L' = 3l^2 \text{ and } \lambda' = \quad \frac{1}{L^2} = -\frac{1}{l^6}.$$

$$\therefore \frac{d\lambda}{dl} = \frac{d\frac{1}{l^3}}{dl} = -\frac{3}{l^4}.$$

* See Classified List, I. 7 and II. 9.
† See Classified List, III. A. 2.

Written inversely,

$$\int \frac{dl}{l^4} = -\frac{1}{3l^3}.*$$

Again, let

$$L = l^3 \text{ and } \lambda = L^3 \quad \therefore \lambda = l^9$$
$$L' = 3l^2 \qquad \lambda' = 3L^2 = 3l^6$$
$$\therefore \frac{d\lambda}{dl} = \frac{dl^9}{dl} = 9l^8.$$

Written inversely,

$$\int l^8 dl = \frac{l^9}{9}.*$$

Again, let

$$L = l^2 \text{ and } \lambda = L^{4\frac{1}{2}} \quad \therefore \lambda = l^9.$$

From the last example $\frac{d\lambda}{dl} = 9l^8$, and also $L' = 2l$. Therefore

$$9l^8 = \lambda'. 2l$$

$$\text{or } \lambda' = \frac{dL^{4\frac{1}{2}}}{dL} = 4\tfrac{1}{2}l^7 = 4\tfrac{1}{2}L^{3\frac{1}{2}}.$$

Similarly, of course, if $X = x^{4\frac{1}{2}}$, then

$$X' = 4\tfrac{1}{2}x^{3\frac{1}{2}}.$$

Written inversely $\int x^{3\frac{1}{2}} dx = \frac{x^{4\frac{1}{2}}}{4\frac{1}{2}}.*$

Again, let

$$L = \cos l \text{ and } \lambda = L^2 = \cos^2 l.$$

Then $\qquad L' = -\sin l \text{ and } \lambda' = 2L = 2\cos l$

$$\therefore \frac{d\cos^2 l}{dl} = -2\cos l \sin l = -\sin 2l.$$

Take two more examples: namely,

$$L = \sin l \text{ and } L = \cos l, \text{ while } \lambda = \frac{1}{L}.$$

$$\text{Then } \lambda' = -\frac{1}{L^2} \text{ in both cases, and}$$

$$L' = \cos l \text{ in } 1^{st} \text{ case}$$
$$= -\sin l \text{ in } 2^{nd} \text{ case.}$$

* See Classified List, III. A. 2.

Therefore, since $1/\sin l = \operatorname{cosec} l$, and $1/\cos l = \sec l$,

$$\frac{d \operatorname{cosec} l}{dl} = -\frac{\cos l}{\sin^2 l}$$

$$\text{and } \frac{d \sec l}{dl} = \frac{\sin l}{\cos^2 l}.$$

86. Reciprocal of a Function.—The second and the last of these illustrations are special cases of the semi-special semi-general case—a very important one—

$$\lambda = \frac{1}{\mathrm{L}}.$$

Here we obtain $\frac{d\lambda}{dl} = \lambda' \mathrm{L}' = -\frac{\mathrm{L}'}{\mathrm{L}^2}$.

Dividing by λ, that is, multiplying by L, we obtain this in the more symmetrical form $\frac{1}{\lambda}\frac{d\lambda}{dl} = -\frac{\mathrm{L}'}{\mathrm{L}}$.

87. Product of two Functions.—Retaining the notation of the last two articles, one particular function, to which we may apply the rule of § 84, is the product λL. Thus

$$\frac{d(\lambda \mathrm{L})}{dl} = \frac{d(\lambda \mathrm{L})}{d\mathrm{L}} \cdot \frac{d\mathrm{L}}{dl}.$$

But by § 80,

$$\frac{d(\lambda \mathrm{L})}{d\mathrm{L}} = \lambda + \mathrm{L}\frac{d\lambda}{d\mathrm{L}}.$$

Multiplying by $\frac{d\mathrm{L}}{dl}$, and observing that by § 84, $\frac{d\lambda}{d\mathrm{L}} \cdot \frac{d\mathrm{L}}{dl} = \frac{d\lambda}{dl}$, there results

$$\frac{d(\lambda \mathrm{L})}{dl} = \lambda\frac{d\mathrm{L}}{dl} + \mathrm{L}\frac{d\lambda}{dl}.$$

If X, Ξ and x be used for the three mutually dependent variables, instead of the letters λ, L and l; and if X′ and Ξ′ be the x-gradients of X and Ξ; then the above is written

$$\frac{d}{dx}(\mathrm{X}\Xi) = \mathrm{X}\Xi' + \mathrm{X}'\Xi.$$

This extremely useful result may be easily proved directly by the method of fig. 17, drawing a curve with ordinates X and Ξ, and considering the increment of the rectangular area $X\Xi$. This increment is evidently

$$X.\delta\Xi + \Xi.\delta X$$

and dividing this by δx the above result is obtained.

Written as an integration this result is

$$\int X\Xi' dx + \int X'\Xi dx = X\Xi$$

$$\text{or} \quad \int X\Xi' dx = X\Xi - \int X'\Xi dx.^*$$

This latter form is the most fundamental and useful of the "formulas of transformation," and is usually referred to as "Integration by Parts." By its help most of the "formulas of reduction" are obtained.

88. Product of any Number of Functions.—This result may at once be extended to the product of any number of different functions Ξ, Ψ, etc. etc. of x. The x-gradient of this product is the sum of a number of terms, each of which is the product of all but one of the factors multiplied by the x-gradient of that one factor omitted. The result may be written in more symmetrical form if each term be divided by the product of all the factors. Thus, call the whole product X, or let

$$X = \Xi \cdot \Psi \text{ etc., etc.}$$

then

$$\frac{X'}{X} = \frac{\Xi'}{\Xi} + \frac{\Psi'}{\Psi} + \text{etc.} + \text{etc.}$$

where X', Ξ', Ψ', etc., are the x-gradients of X, Ξ, Ψ, etc.

89. Reciprocal of Product of Two Functions.—The x-gradient of the reciprocal of the product of two functions Ξ and Ψ of x is found with equal ease. Call this reciprocal X and its x-gradient X'. Then

$$X = \frac{1}{\Xi} \cdot \frac{1}{\Psi}$$

and by last paragraph, § 88 and § 86

$$\frac{X'}{X} = \Xi\left(-\frac{\Xi'}{\Xi^2}\right) + \Psi\left(-\frac{\Psi'}{\Psi^2}\right)$$

$$= -\frac{\Xi'}{\Xi} - \frac{\Psi'}{\Psi}.$$

* See Classified List, I. 8.

90. Reciprocal of Product of any Number of Functions.—The extension of this law to the reciprocal of the product of any number of functions is made by simply repeating the process of the last paragraph.

91. Ratio of Two Functions.—To find the x-gradient of the quotient of two functions of x, we combine the laws of § 87 and § 86. Thus, if

$$X = \frac{\Xi}{\mathfrak{X}} = \Xi \cdot \frac{1}{\mathfrak{X}}$$

then

$$\frac{X'}{X} = \frac{\Xi'}{\Xi} + \mathfrak{X}\left(-\frac{\mathfrak{X}'}{\mathfrak{X}^2}\right) = \frac{\Xi'}{\Xi} - \frac{\mathfrak{X}'}{\mathfrak{X}}.$$

It will be a useful exercise for the student to deduce this result directly by the method of fig. 18 and § 81, making the co-ordinates to the curve Ξ and $\mathfrak{X}$ instead of X and x, and without using the law $\frac{d\lambda}{dl} = \lambda' L'$.

92. Ratio of Product of any Number of Functions to Product of any Number of other Functions.—It is now apparent that all the results of §§ 86–91, inclusive, may be combined into the following single more general formula :—if

$$X = \frac{L\,M\,N, \text{ etc., etc.}}{P\,Q\,R, \text{ etc., etc.}}$$

where L, M, N, P, Q, R, etc., are functions of x, and if X′, L′, P′, etc., be the x-gradients; then

$$\frac{X'}{X} = \frac{L'}{L} + \frac{M'}{M} + \frac{N'}{N} + \text{ etc., etc.} - \frac{P'}{P} - \frac{Q'}{Q} - \frac{R'}{R} - \text{ etc., etc.}$$

93. Theory of Resultant Error.—If the last formula be multiplied by δx, and if we call $X'\delta x \equiv \delta X$, $L'\delta x \equiv \delta L$, $P'\delta x \equiv \delta P$, etc.; all direct reference to x disappears and the formula becomes

$$\frac{\delta X}{X} = \frac{\delta L}{L} + \frac{\delta M}{M} + \frac{\delta N}{N} + \text{ etc. etc.} - \frac{\delta P}{P} - \frac{\delta Q}{Q} - \frac{\delta R}{R} - \text{ etc., etc.}$$

Now, if L, M, etc., represent measurements of physical quantities that have been made in order to calculate from them the quotient X; and if, by any means, it be known or estimated that the measurement of L has been subject to a small error δL, and that the measurement of M has been subject to a small error δM, etc., etc.; then $\frac{\delta L}{L}$ is the **ratio of error** in the measurement of L, and $\frac{\delta M}{M}$ is the **ratio of error** in the measurement of M, and similarly with the other factors; while $\frac{\delta X}{X}$ is the **resulting ratio of error** in the **calculated** quotient X.

The above proposition may then be expressed in the following words:—

The ratio of the product of a number of measured quantities to the product of another set of measured quantities is subject to a ratio of error, equal to the sum of the ratios of error in the individual multipliers reduced by the sum of the ratios of error in the individual divisors.

Attention must here be paid to the **signs** of the errors. Thus, if δL is a negative error, then $\frac{\delta L}{L}$ is really a negative fraction. Similarly, if δP is a negative error, then $\left(-\frac{\delta P}{P}\right)$ is really a positive fraction.

Now, although there is often reason for supposing the suspected error to lie more probably in one direction than in the opposite, still errors being things which are avoided as far as possible (or convenient, or profitable), and, therefore, not consciously incurred, it is never known for *certain* whether any individual error be + or −. Thus, although divisors give by the formula ratios of error of opposite sign to those given by multipliers, it *may* happen that all the terms in $\frac{\delta X}{X}$ are really of the same sign, and have, therefore, all to be arithmetically added to get the whole ratio of error. Thus, if we are considering what may be the **maximum possible** error in X, we must pay no attention to the difference between multipliers and divisors, but add all the ratios of error in all the factors (both multipliers and divisors) together independently of sign.

It need hardly be pointed out that this **maximum possible** error is greater than the **probable** error.

94. Exponential Function.—In fig. 20 is drawn a curve with horizontal ordinates called l, and vertical ordinates b^l. l varies continuously. Here l is essentially a number; not, of course, necessarily a whole number, since its variation is continuous. b is a constant, and receives the name of the "**base**" of this curve. If b were any physical quantity, then its different powers would have various physical "dimensions," and would mean physical quantities of different **kinds**. In fig. 20 we assume the various vertical ordinates as all of the same kind, and therefore b cannot be a physical quantity, but must be a pure number. On this assumption we find that b^l is also a pure number. In this problem, therefore, both horizontal and vertical ordinates can be nothing but pure numbers.*

At the same time it must be noted that if k be a physical constant quantity, either a unit or any other quantity, then kb^l represents a varying physical quantity of the same kind.

95. Power Gradient of Exponential Function.—If n be any **fixed** number, b^n is also a **fixed constant** number. We may write

$$b^l = b^n . b^{(l-n)}.$$

Here b^l is the vertical height of the curve, fig. 20, at any horizontal distance l; while $b^{(l-n)}$ is the height of a point on the curve at the **constant** horizontal distance n to the left of l.

Considering various l's and various pairs of points with ordinates l and $(l-n)$, we perceive that the nature of this curve gives a constant proportion, b^n, between the heights of all pairs of points at the constant horizontal distance n apart. A succession of points at the equal horizontal spacing n have their heights advancing in geometrical series, the common ratio of which is b^n.

Taking the l-gradient, or slope, of the curve at l, and remembering that b^n is a constant factor, we find

$$\frac{db^l}{dl} = b^n . \frac{db^{(l-n)}}{dl}.$$

This means that there is also the same constant proportion, b^n, between the gradients as between the heights at all pairs of points horizontally n apart.

Dividing the height by the slope we obtain the subtangent, see § 38.

$$\text{Subtangent at } l = \text{say } \mathrm{T} = b^l \Big/ \frac{db^l}{dl}.$$

$$\text{Subtangent at } (l-n) = \text{say } \mathrm{T}_{(l-n)} = b^{(l-n)} \Big/ \frac{db^{(l-n)}}{dl}.$$

These two last expressions are equal, because the common factor b^n cancels out in their ratio. Thus the two subtangents are equal in length; see fig. 20.

Now the length of the subtangent at l does not depend in any way upon the length n. For any one point l, we may take various lengths n, thus getting various points $(l-n)$, at **all of which** the subtangent equals that at the one point l. This means that **at all points along the curve** the subtangent has the same length.

* In Sir William Hamilton's *Quaternions*, (-1) raised to a continuously increasing power is used to indicate rotation or gradual change of direction.

Inverting the last equation,

$$\frac{db^l}{dl} = \frac{1}{T}.b^l = \text{Constant} \times b^l,$$

the constant being the reciprocal of the constant length of subtangent. The subtangent being a pure number, its reciprocal is also a pure number.

96. Natural, Decimal, and other Logarithms.—When $T = 1$ (unity), and the above constant is therefore also unity, the base b is designated by mathematicians by the letter e. It, e, is the base of the "**natural**" or "**hyperbolic**" or "**Neperian**" logarithms.

Each system of logarithms has a "**base**," which is a number and to which all other numbers are referred. Every number, whether whole or fractional, is represented as this "base," raised to a certain power. The power to which the base must be raised in order to produce each number is the logarithm of that number.

Thus, if b^l be a number, and if b be taken as the base of the system of logarithms, then l is the log of the number b^l to the base b.

In the "**Common,**" or "**Brigg's,**" or "**Decimal**" logarithms, the base is 10; every number being represented as 10 raised to one or other power.

The result of § 95 may be expressed thus:—In any system of logarithms the logarithm-gradient of the number bears a constant proportion to the number itself. The reciprocal of this constant proportion, or the subtangent in the graphic representation of fig. 20, is called the "**modulus**" of the system of logarithms.

The "natural" system of logs may be defined as that which gives the rate of increase of the number compared with its logarithm equal to the number itself; or which makes the logarithm-gradient of the number equal to the number itself; or, symbolically,

$$\frac{de^l}{dl} = e^l$$

the constant for this base e being unity. The base e is calculated to be 2·71828 - - - -. The modulus of this system is 1.

97. Number Gradient of Logarithm.—If we call the number N, then in fig. 20 the vertical height of the curve is N and the horizontal ordinate is its logarithm. The curve is sometimes termed a logarithmic curve; sometimes an exponential curve. It takes different forms according to the logarithmic base b used in drawing it out. In fig. 21 these variations of form are shown, the five curves having the bases

$$b = 2,\ e,\ 8,\ 10,\ \text{and}\ 12.$$

All these curves come to the same height $N=1$ at $l=0$. At $l=1$ the height of each is the base b. To the left of the vertical axis where l is negative, the height N is always less than unity, and decreases asymptotically to the horizontal axis towards zero height. Fig. 22 shows the extension of these same curves to high numbers with positive logarithms. The logarithm to the base e of the number N is written $\log_e N$. The logarithm to the base b of N is written $\log_b N$. Thus the decimal log of N may be unambiguously written $\log_{10} N$.

The results of §§ 95 and 96 written in this notation, and taking the reciprocal or conjugate gradient from the vertical axis, instead of the l-gradient, are

$$\frac{d \log N}{dN} = \frac{dl}{dN} = \frac{T}{N},$$

the constant T being the modulus of the system of logarithms; and

$$\frac{d \log N}{dN} = \frac{dl}{dN} = \frac{1}{N}$$

if natural logs are used in which the base is e and the modulus unity.

98. Relation between Different Log "Systems."—Any base b may be looked on as the base e raised to a certain power, which we will call $\frac{1}{T}$. Thus

$$b = e^{\frac{1}{T}} \text{ and } e = b^T.$$

Therefore, if

$$N = b^l = e^{\frac{l}{T}};$$

also

$$\log_b N = l = T \log_e N = \log_e N \,.\, \log_b e.$$

Thus there is a constant ratio $T = \log_b e$ between the logarithms of numbers to the base b to their logarithms in the "natural" system.

Taking the N-gradients of these last logs and remembering that T is a constant,

$$\frac{d \log_b N}{dN} = T \frac{d \log_e N}{dN} = \frac{T}{N}$$

which shows that the T of this paragraph means the same as the T of the last, § 97, or the subtangent of fig. 20.

Written as an integration result, this is

$$\int \frac{dN}{N} = \frac{\log_b N}{\log_b e} = \log_e N.*$$

It is useful to notice that $a^{\log_b x} = x^{\log_b a}$, because taking logs on each side, we find $\log_b x \,.\, \log_b a = \log_b a \,.\, \log_b x$.

It follows that $\int a^{\log_b x} dx = \frac{x^{1+\log_b a}}{1+\log_b a}$.†

99. Base of Natural Logs.—Although the calculation of e, or of the modulus T of any logarithmic system, is very laborious, there is no other difficulty about it beyond its tediousness. Thus to find the modulus of, say, the decimal system, an extremely small root of 10 has to be extracted. Thus the square root may be extracted, say, 100 times over. This will give $(\frac{1}{2})^{100}$th power of 10, which is a number a very minute fraction over 1. Now the decimal logarithm of this number is $(\frac{1}{2})^{100}$, which is easy to calculate, and the logarithm of 1 is zero. The former is, therefore, the increment of the logarithm corresponding to the number increment from 1 to the $(\frac{1}{2})^{100}$th power of 10. If the ratio of this logarithm increment to the number increment be multiplied by the number, which is here 1, the product is the decimal log-modulus.

The laborious part of the operation is finding the $(\frac{1}{2})^{100}$th power of 10. Extracting only a higher root, the increments will be larger, and the calculation will not give so great accuracy. Thus, if the extraction of the square root be repeated only 10 times, the result of the calculation gives

$$T_{10} = 0{\cdot}434\ 294\ 116\ ‡$$

whereas by more minute calculation its true value to 9 decimal places has been found to be

$$0{\cdot}434\ 294\ 482.$$

The error is only 0·000 000 366.

100. Logarithmic Differentiation.—Taking again the function of § 92, and taking logs, we have

$$\log X = \log L + \log M + \text{etc.} - \log P - \text{etc.}$$

* See Classified List, III. A. 3. † See Classified List, IV. 7.

‡ Calculated by Mr R. F. Muirhead from the value

$$\frac{\log 10^{2^{-10}} - \log 10^{-2^{-10}}}{10^{2^{-10}} - 10^{-2^{-10}}} = \frac{10^{2^{-10}}}{2^9\left(10^{2^{-9}} - 1\right)}.$$

and differentiating with respect to x, we have by §§ 84 and 98

$$\mathrm{T}\frac{\mathrm{X}'}{\mathrm{X}}=\mathrm{T}\left\{\frac{\mathrm{L}'}{\mathrm{L}}+\frac{\mathrm{M}'}{\mathrm{M}}+\text{etc.}-\frac{\mathrm{P}'}{\mathrm{P}}-\text{etc.}\right\}$$

which is the same result again as was found in § 92.

This method of differentiating products of functions is called "**logarithmic differentiation.**"

101. Change of the Independent Variable.—The rule deduced in § 80 was written

$$\int \mathrm{X}dx=\mathrm{X}x-\int \mathrm{X}'xdx.$$

Now $\mathrm{X}'\delta x$ is the same as $\delta\mathrm{X}$; so that the integral on the right hand may be written $\int xd\mathrm{X}$. The equation becomes then

$$\int \mathrm{X}dx=\mathrm{X}x-\int xd\mathrm{X}.$$

If $\int xd\mathrm{X}$ is easier to find than $\int \mathrm{X}dx$, this forms a method of facilitating the latter integration. Such a transformation is called a "**change of the independent variable**" or "**substitution.**"

Fig. 23 is the graphic representation of this law. Taking the

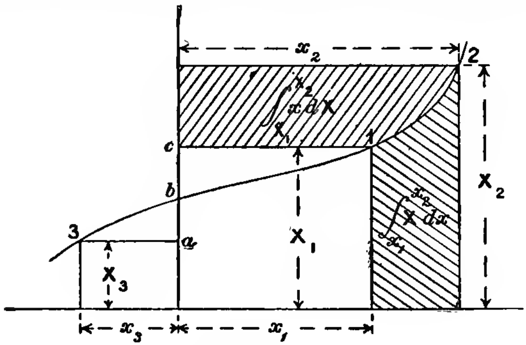

FIG. 23.

integration between the points 1 and 2 of the curve $x\mathrm{X}$, the transformation is written

$$\int_{x_1}^{x_2}\mathrm{X}dx=(\mathrm{X}_2x_2-\mathrm{X}_1x_1)-\int_{\mathrm{X}_1}^{\mathrm{X}_2}xd\mathrm{X}.$$

In fig. 23 the sum of the two areas rafined over thus ///// and ||||| is evidently $(X_2x_2 - X_1x_1)$. The first of these areas, namely, that between the curve, the vertical axis, and the two horizontal lines at levels X_1 and X_2, is $\int_{X_1}^{X_2} x dX$. The second, included between the curve, the horizontal axis, and the two verticals at distances x_1 and x_2, is $\int_{x_1}^{x_2} X dx$. Hence the above equation.

In fig. 23 the point 3 has a negative ordinate x, while its X-ordinate is positive. A useful exercise for the student is to follow the variations of the proposition, taking the pair of points in each possible pair of quadrants of the full diagram.

CHAPTER V.

PARTICULAR LAWS.

102. Any Power of the Variable.—In § 99 the integral of $\frac{1}{x}$ was found to be $\log_e x$. This is the single exception to a general law giving the integral of any power of a variable, special examples of which have already been demonstrated in §§ 59, 61, 71, 79, and 85, namely, the integrals of the 0^{th}, 1^{st}, 2^{nd}, $(-2)^{nd}$, 5^{th}, $(-4)^{th}$, 8^{th}, and $3\frac{1}{2}^{th}$ powers.

All these examples conform to the general law

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \text{ *}$$

where C is the integration constant;

$$\text{or } \frac{dx^{n+1}}{dx} = (n+1)x^n .$$

103. Any Power of the Variable by Logarithms.—This result can be proved to be always true by logarithmic differentia-

* See Classified List, III. A. 2.

tion as explained in § 100; thus, n being any power, positive or negative, integer or fractional,

$$X = x^n$$

$$\therefore \log X = n \log x$$

$$\therefore \frac{X'}{X} = \frac{n}{x}$$

$$\therefore X' = n\frac{X}{x} = nx^{n-1}$$

104. Diagram showing Integral of x^{-1} to be no Real Exception.—This formula is true for all powers of the variable, with the one exception of the integral of $\frac{1}{x}$ or x^{-1}, which is $\log_e x$.

It is often a puzzling question to students why there should be this one solitary exception to so general a law. Fig. 24 has been drawn to demonstrate graphically that it is only *formally* an exception; that it is in reality no exception at all. To compare this one apparent exception with other cases of the general law, the integrations must be taken between the same limits. It may be convenient to take the lower limit of x equal to 1 because $\log 1 = 0$ and $\Big[\log x\Big]_1^x = \log x$. Taking the same lower limit for $\frac{1}{n}x^n$ we have $\frac{1}{n}\Big[x^n\Big]_1^x = \frac{x^n - 1}{n}$ because $1^n = 1$ whatever n be.

In fig. 24 there are drawn to the scales shown, the curves $\frac{x^n - 1}{n}$ for $n = 2, 1, \frac{1}{2}, \frac{1}{10}, -\frac{1}{10}, -\frac{1}{2}, -1$, and -2. There is also plotted to the same scales the curve $\log_e x$. It will be seen that the curves for $n = \frac{1}{10}$ and $n = -\frac{1}{10}$ lie very close together, and that the curve $\log_e x$ lies between them throughout its whole length. This shows that the logarithmic curve is simply one of the general set of curves illustrating the general law, and that it is no real exception to the general law. Its position between the curves for $n = \pm\frac{1}{10}$ shows that $\log_e x$ is simply the special name given to the value of the function $\frac{x^n - 1}{n}$ when n is an excessively minute fraction, or rather when n is zero. Considering the variation of the curve in fig. 24 downwards from positive values of n to negative values of n, it is clear that the curve must have some definite position as n passes through zero, a position lying between that for small positive values of n and small negative values of n. This position is that of the curve $\log_e x$. For $n=0$, the function $\frac{x^n-1}{n}$ takes the indeterminate form $\frac{x^0-1}{0}=\frac{1-1}{0}=\frac{0}{0}$, and its value has to be found by a special method, the result appearing in a special form. It should be noted that all the curves pass through the height 0 at the horizontal distance $x=1$, and that they have here one common tangent or gradient $=1$.

105. Any Power of Linear Function.—If a, b, and n are constants, and

$$X=(a+bx)^n,$$

we have by § 84 and last article,

$$X'=\frac{dX}{d(a+bx)}\times\frac{d(a+bx)}{dx}=n(a+bx)^{n-1}\times b=bn(a+bx)^{n-1}.$$

Written inversely for integration, this is

$$\int(a+bx)^n dx=\frac{1}{b(n+1)}(a+bx)^{n+1}+C\ *$$

the constant C being introduced by the integration.

106. Reciprocal of any Power of Linear Function.—This last integration rule fails when $n=-1$.

In this case we find by §§ 51 and 98,

$$\int(a+bx)^{-1}dx=\frac{1}{b}\log_e(a+bx)+C=\frac{2\cdot3\text{---}}{b}\log_{10}(a+bx)+C.\dagger$$

107. Ratio of Two Linear Functions.—The function $\frac{ax}{b+cx}$ can be reduced so as to make it depend on the last case,

$$\text{because } \frac{ax}{b+cx}=\frac{a}{c}-\frac{\frac{ab}{c}}{b+cx}.$$

Therefore, by §§ 102 and 106,

$$\int\frac{ax}{b+cx}dx=\frac{ax}{c}-\frac{ab}{c^2}\log_e(b+cx)+C.$$

108. Ratio of Two Linear Functions; general case.—Since the function

$$\frac{A+Bx}{a+bx}=\frac{A}{a+bx}+\frac{Bx}{a+bx},$$

* See Classified List, III. A. 4. † See Classified List, III. A. 5.

the integration of this function is performed by combining the results of §§ 106 and 107.*

109. Quotient of Linear by Quadratic Function. — If $X = a + bx^2$; then $X' = 2bx$ and $\frac{x}{a+bx^2} = \frac{1}{2b}\frac{X'}{X}$.

Now $\int \frac{X'}{X} dx = \int \frac{dX}{X} = \log_e X$ by § 98;

therefore

$$\int \frac{x}{a+bx^2} dx = \frac{1}{2b} \log_e (a+bx^2) + C$$

where C is the integration constant.

Similarly if $X = a + bx + cx^2$; then $X' = b + 2cx$, and, therefore, any function of the form

$$\frac{A + Bx}{a + bx + cx^2}$$

can be readily integrated by splitting it into two terms as in § 107.†

110. Indicator Diagrams.—An important case of the use of the law of §§ 102 and 105 is the integration of the work measured by an indicator diagram.

If at any stage of the expansion p be the pressure and v be the volume of the working substance, then as the volume increases by dv, the work done is pdv.

Taking the expansion law in the more general form of § 105, or

$$p = (a + bv)^{-n};$$

then the work done during expansion from p_1, v_1 to p_2, v_2 is

$$\text{Expansion work done } W = \int_1^2 pdv = \int_1^2 (a+bv)^{-n} dv$$

$$= \frac{1}{b(1-n)} \Big[(a+bv)^{-n+1} \Big]_1^2$$

$$= \frac{1}{b(1-n)} \Big[p(a+bv) \Big]_1^2 = \frac{1}{b(n-1)} p_1^{\frac{n-1}{n}} \left\{ 1 - \left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}} \right\}.$$

Here the index is always negative. If it is arithmetically greater than 1, the expansion curve makes $\Big[p(a+bv) \Big]_1^2$ negative. But at the same time the divisor $(1-n)$ is negative, so that the formula makes the work done positive. It is then better to reverse the limits and to use the positive divisor $(n-1)$.

* See Classified List, III. A. 6. † See Classified List, III. A. 17.

If $a=0$, or $p=bv^{-n}$, as in most approximate formulas for expansion curves, the result simplifies, by cancelling out b from numerator and divisor, to*

$$W=\frac{1}{n-1}\Big[pv\Big]_2^1$$

$$=p_1v_1\frac{1-\frac{p_2}{p_1}\cdot\frac{v_2}{v_1}}{n-1}$$

$$=p_1v_1\frac{1-\left(\frac{v_1}{v_2}\right)^{n-1}}{n-1}$$

$$=p_1v_1\frac{1-\left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}}}{n-1}$$

$$=p_2v_2\frac{\left(\frac{v_2}{v_1}\right)^{n-1}-1}{n-1}$$

$$=p_2v_2\frac{\left(\frac{p_1}{p_2}\right)^{\frac{n-1}{n}}-1}{n-1}.$$

These formulas, which are all practically useful, give the work done during expansion in terms of the ratios between the initial and final volumes, and of the initial and final pressures; also in terms of the initial product pv and of the final product pv. The latter formula is most useful in the case of air and gas compression pumps where the initial and known volume and pressure are v_2p_2.

The "**admission**" part of the indicator diagram has an area p_1v_1, and this has to be added to the above, giving the total work done

$$W_t=p_1v_1\frac{n-\left(\frac{v_1}{v_2}\right)^{n-1}}{n-1}$$

$$=p_1v_1\frac{n-\left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}}}{n-1}.$$

These calculations do not take account of the back pressure deduction from the area of the card.

* The constant b used here equals the $-n^{th}$ power of the b used in the previous formula.

The "mean pressure" of this total area is the last value of W divided by v_2, or

$$\frac{p_m}{p_1} = \left\{ \frac{v_1}{v_2} \cdot \frac{n - \left(\frac{v_1}{v_2}\right)^{n-1}}{n-1} \right\}.$$

From this the back pressure must be subtracted to obtain the "effective" mean pressure.

In the case of isothermal gas expansion, $n = 1$ or $pv = b$, and the integration for work done during expansion is

$$W = b\Big[\log_e v\Big]_1^2 = 2{\cdot}3 p_1 v_1 \log_{10} \frac{v_2}{v_1} = 2{\cdot}3 p_1 v_1 \log_{10} \frac{p_1}{p_2}$$

and including the work during admission

$$W_t = p_1 v_1 \left\{ 1 + 2{\cdot}3 \log_{10} \frac{v_2}{v_1} \right\}.$$

The ratio of mean to initial pressure is therefore

$$\frac{p_m}{p_1} = \frac{v_1}{v_2} \left\{ 1 + 2{\cdot}3 \log_{10} \frac{v_2}{v_1} \right\}.$$

111. Graphic Construction for Indicator Diagrams.—In fig. 25 the upper curve is a common hyperbola or curve of reciprocals,

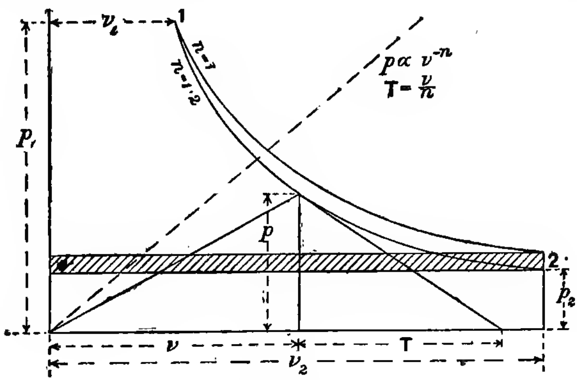

FIG. 25.

and is the gas isothermal. The lower is drawn to the formula $p = bv^{-1{\cdot}2}$. The product pv is the same at all points of the upper curve, and, therefore, at all points equals $p_1 v_1$. Therefore for the point 2 on the lower curve, the horizontal strip of area rafined over equals $(p_1v_1 - p_2v_2)$; and this divided by $n-1=\cdot 2$, *i.e.*, multiplied by 5, equals the work done under the lower curve during the expansion from 1 to 2.

The mean pressure, including the admission period, therefore, equals 5 times the height of the strip rafined over *plus* the height to the upper edge of the same strip.

The gradient of the curve $p=bv^{-n}$ is negative, and equals $p'=-nbv^{-n-1}=-n\frac{p}{v}$. Therefore $n=p'\times\frac{v}{p}$ omitting the *minus* sign which only indicates that the forward slope is downwards. But if T be the subtangent, then $p'=\frac{p}{\mathrm{T}}$. Therefore we find $n=\frac{v}{\mathrm{T}}$, and $\frac{1}{n-1}=\frac{\mathrm{T}}{v-\mathrm{T}}$. Thus in investigating actual indicator cards taken from engines or compressing pumps, at each point of the expansion curve at which a fair tangent can be accurately drawn, the value of the index n can be found by measuring the ratio of v to T. Also in finding the mean pressure by adding to the height of the upper edge of the rafined strip of fig. 25 the depth of this strip divided by $(n-1)$, this division can be performed very easily by an evident graphic construction, since $\frac{1}{n-1}=\frac{\mathrm{T}}{v-\mathrm{T}}$.

Conversely, in constructing theoretical indicator diagrams, when a few points of the curve have been calculated, it much assists in the fair drawing in of the curve to draw the tangents at these points, which can easily be done by setting off for each point $\mathrm{T}=\frac{v}{n}$. If an oblique line be drawn at a tangent of inclination n to the vertical axis (it is drawn dotted in fig. 25), then at each v the height of this line will give the corresponding T. In fact, by this construction the whole curve may be accurately drawn out from point to point by drawing a connected chain of short tangents whose direction is at each point obtained in this way; the accuracy of the construction being very considerable if care be taken that each short tangent length shall stretch equally behind and in front of the point at which its direction is found by plotting T. By this construction the labour of logarithmic calculation of the heights of a series of points is rendered unnecessary.*

112. Sin^{-1}x and $(r^2-x^2)^{-\frac{1}{2}}$.—In § 74 it was found that the angle-gradient of a sine is the cosine, and that of the cosine *minus* the

* See Appendix E for further information concerning this class of curve.

sine. That is, if a be the angle and s its sine, or $\sin a = s$; then since $\cos^2 a = 1 - s^2$, we have

$$\frac{ds}{da} = (1 - s^2)^{\frac{1}{2}}.$$

When the angle is measured by its sine it is symbolically expressed as $\sin^{-1}s \equiv$ "the angle whose sine is s." Using this notation, and taking the reciprocal of the above; *i.e.*, taking the cogradient or the "sine-gradient of the angle," we have

$$\frac{d \sin^{-1}s}{ds} = \frac{1}{(1 - s^2)^{\frac{1}{2}}}.$$

From this we deduce the more general result

$$\frac{d\left\{a \sin^{-1}\frac{s}{r}\right\}}{ds} = \frac{a}{(r^2 - s^2)^{\frac{1}{2}}}$$

where a and r are constants.

The corresponding integrations are, when x instead of s is used to indicate the variable,

$$\int \frac{a\,dx}{(r^2 - x^2)^{\frac{1}{2}}} = a \sin^{-1}\frac{x}{r} + C$$

$$= C' - a \cos^{-1}\frac{x}{r}.*$$

The two angles having the same fraction for sine and cosine respectively are complementary; so that these two forms of the integral only differ in the integration constants $\left(C' - C = \frac{\pi a}{2}\right)$ and in the sign of the variable parts.

The sine of an angle cannot be greater than $+1$ nor less than -1. These integration formulæ would have, therefore, no meaning in cases in which $x > r$ or $x < -r$. These limits correspond with those within which $(r^2 - x^2)^{\frac{1}{2}}$ remains real, because the square root of a negative quantity is "**impossible**" or "**imaginary.**" If $(r^2 - x^2)^{\frac{1}{2}}$ arises from any actual physical problem, such a problem can never throughout the whole actual range of x make $x > r$.

113. $(1 - x^2)^{\frac{1}{2}}$ integrated or Area of Circular Zone.—In § 59 it was shown that the area of a sector of a circle equals $\frac{1}{2}r^2 a$. The

* See Classified List, III. B. 6 and 5.

angle may be expressed in terms of its sine as in last article. If s be the sine, we have

$$\text{Sectorial area} = \tfrac{1}{2}r^2 \sin^{-1}s.$$

In fig. 13 this is the area NaO; in which figure the length as measures rs of the present article, and $ac = r\cos a = r\sqrt{1-s^2}$ of the present article. The triangular area aOc, therefore, equals $\frac{1}{2}r^2 s\sqrt{1-s^2}$. Add this to the above sectorial area; the sum is the area ONac. This area may be taken as made up of a large number of narrow strips parallel to ON, the height of each of which would be $r\cos a = r\sqrt{1-s^2}$, while the horizontal width would be $r.ds$. The area ONac is, therefore, the integral of this narrow strip of area from $a = o$ to $a = a$, which limits correspond to from $s = o$ to $s = s$. Thus

$$\int r\sqrt{1-s^2}\cdot r ds = r^2\int\sqrt{1-s^2}ds = \tfrac{1}{2}r^2 s\sqrt{1-s^2} + \tfrac{1}{2}r^2\sin^{-1}s$$

or

$$\int(1-s^2)^{\frac{1}{2}}ds = \frac{s(1-s^2)^{\frac{1}{2}} + \sin^{-1}s}{2}.*$$

Twice this is the area of a circular zone lying between a diameter and a parallel at the height s from the diameter, the radius being assumed 1 in the last equation.

Here, again, s cannot range outside the limits ± 1.

114. $x(r^2-x^2)^{-\frac{1}{2}}$ **integrated.**—The function $x(r^2-x^2)^{-\frac{1}{2}}$ may be looked on as the sine divided by the cosine, *i.e.*, the tangent of an angle, see fig. 13, while dx is the increment of the sine. The increment of the sine multiplied by the tangent evidently equals the decrement of the cosine, and accordingly the integral is *minus* the cosine, or $-(r^2-x^2)^{\frac{1}{2}}$.†

115. $(x^2 \pm r^2)^{-\frac{1}{2}}$ **integrated.**—The function $(x^2 \pm r^2)^{-\frac{1}{2}}$ is more difficult to deal with. Let X represent any function of x, and multiply and divide its reciprocal by $(x + \mathrm{X})$; thus:—

$$\int\frac{dx}{\mathrm{X}} = \int\frac{1}{\mathrm{X}}\cdot\frac{x+\mathrm{X}}{x+\mathrm{X}}dx = \int\frac{1+\frac{x}{\mathrm{X}}}{x+\mathrm{X}}dx$$

$$= \mathrm{C} + \log_e(x+\mathrm{X}) \text{ if } \mathrm{X}' = \frac{x}{\mathrm{X}}.$$

The condition $\mathrm{X}' = \dfrac{d\mathrm{X}}{dx} = \dfrac{x}{\mathrm{X}}$ gives $xdx = \mathrm{X}d\mathrm{X}$

* See Classified List, III. B. 9.

† See Classified List, III. B. 7. Note also that, since $x = -\frac{1}{2}\frac{d}{dx}(r^2-x^2)$, therefore $x(r^2-x^2)^{-\frac{1}{2}}$ can be recognised directly, by § 84, to be the x-gradient of $-(r^2-x^2)^{\frac{1}{2}}$.

or integrating

$$x^2 + k = X^2 \text{ or } X = (x^2 + k)^{\frac{1}{2}}$$

where the integration constant k may be either + or −. Writing $k = \pm r^2$, we have

$$\int \frac{dx}{(x^2 \pm r^2)^{\frac{1}{2}}} = C + \log_e\{x + (x^2 \pm r^2)^{\frac{1}{2}}\}.*$$

Here, if k is negative, the differential is "imaginary" and cannot occur in any physical problem except for values of x greater than $\sqrt{-k}$.

116. $x^{-1}(r^2 - x^2)^{-\frac{1}{2}}$ integrated.—The integral of $x^{-1}(r^2 - x^2)^{-\frac{1}{2}}$ is found most easily by substituting $\frac{1}{X}$ for x. Thus

$$X = \frac{1}{x};\ X' = -\frac{1}{x^2} = -X^2;\ \frac{dx}{x^2} = -dX.$$

Therefore

$$\int \frac{dx}{x(r^2 - x^2)^{\frac{1}{2}}} = \frac{1}{r}\int \frac{dx}{x^2\left(\frac{1}{x^2} - \frac{1}{r^2}\right)^{\frac{1}{2}}} = -\frac{1}{r}\int \frac{dX}{\left(X^2 - \frac{1}{r^2}\right)^{\frac{1}{2}}}$$

$$= C' - \frac{1}{r}\log_e\left\{X + \left(X^2 - \frac{1}{r^2}\right)^{\frac{1}{2}}\right\} \text{ by § 115}$$

$$= C - \frac{1}{r}\log_e \frac{r + (r^2 - x^2)^{\frac{1}{2}}}{rx}.\dagger$$

117. Log x integrated.—The integral of the logarithm of a variable number N is found by help of the formula of reduction in § 80 and by § 98, thus:—

$$\int \log_b N dN = N \log_b N - \log_b e \int \frac{N}{N} dN + C$$

$$= N\{\log_b N - \log_b e\} + C.$$

$\log_b e$ is the "modulus" of the system of logarithms whose base is b, and for the decimal system is 0·4343 nearly. Therefore,

$$\int \log_{10} N dN = N\{\log_{10} N - \cdot 4343\} + C.\ddagger$$

* See Classified List, III. B. 6, 3, and 4.
† See Classified List, III. B. 13 and 10.
‡ See Classified List, IV. 4.

118. Moment and Centre of Area of Circular Zone.—With the notation already used, we saw in § 113 that a narrow strip of the area of a semicircle is $2r^2(1-s^2)^{\frac{1}{2}}ds$. The distance of this strip from the diameter from which the angle and its sine are measured is rs, and the product of the area by this distance is the moment of the strip-area round this diameter. This is $2r^3s(1-s^2)^{\frac{1}{2}}ds$, in which r is a constant while s varies. Since $s = -\frac{1}{2}\frac{d(1-s^2)}{ds}$, the integral moment of all the strips for a zone between the diameter and a parallel rs away from it is easy to find. Calling $(1-s^2)$ by letter S, we find

$$\begin{aligned}\text{Integral moment} &= 2r^3\int s(1-s^2)^{\frac{1}{2}}ds \\ &= -r^3\int S^{\frac{1}{2}}dS \\ &= -\tfrac{2}{3}r^3S^{\frac{3}{2}} \\ &= \tfrac{2}{3}r^3\{1-(1-s^2)^{\frac{3}{2}}\}\end{aligned}$$

when taken from lower limit $s=0$ or $S=1$.

From this is deduced by dividing by the area of the zone;

Distance of centre of area of zone from diameter

$$= \tfrac{2}{3}r\frac{1-(1-s^2)^{\frac{3}{2}}}{s(1-s^2)^{\frac{1}{2}}+\sin^{-1}s}.$$

For the whole half circle, $\sin^{-1}s$ becomes a right angle or $\frac{\pi}{2}$; while $s=1$ and $(1-s^2)=0$. Therefore the centre of area of a semicircle is distant from the centre of the circle by

$$\tfrac{2}{3}r \cdot \frac{1}{\frac{\pi}{2}} = \frac{4}{3\pi}r = \cdot 4244r.$$

The moment of the whole semicircular area round the diameter is $\frac{2}{3}r^3$.

The integration performed here is a geometrical illustration or proof of the general integral of $x(1-x^2)^{\frac{1}{2}}$.*

119. $(r^2+x^2)^{-1}$ **integrated.**—In § 78 it was found that the angle-gradient of the tangent equals the square of the reciprocal of the cosine. Remembering that

$$\cos^2\alpha = \frac{1}{1+\tan^2\alpha}$$

* See Classified List, III. B. 9.

we find

$$d\alpha = \frac{d \tan \alpha}{1 + \tan^2\alpha}.$$

Call $\tan \alpha \equiv t$, and therefore $\alpha = \tan^{-1}t$; we then obtain the integration

$$\int \frac{dt}{1+t^2} = \int d\alpha = \alpha = \tan^{-1}t + C.*$$

Here t is essentially a number or pure ratio, and it may vary from $-\infty$ to $+\infty$. If, in order to make the formula of more general application, we introduce a constant r^2 as follows, then t may be any + or − physical quantity, but t and r must be of the same **kind.** Then, since

$$\frac{d\left(\frac{t}{r}\right)}{dt} = \frac{1}{r}$$

$$\int \frac{dt}{r^2 + t^2} = \frac{1}{r}\tan^{-1}\frac{t}{r} + C.\dagger$$

120. $(r^2 - x^2)^{-1}$ **integrated.**—Since $(r^2 - x^2) = (r + x)(r - x)$, we find easily

$$\frac{1}{r^2 - x^2} = \frac{1}{2r}\left(\frac{1}{r+x} + \frac{1}{r-x}\right)$$

and therefore this function can be integrated by help of § 106.‡

121. Hyperbolic Functions and their Integrals.—The functions $\frac{1}{2}(e^x + e^{-x})$ and $\frac{1}{2}(e^x - e^{-x})$ enter largely into the geometry of the hyperbola and of the catenary, as well as into the investigation of several important stress and kinetic problems. From the origin of their utilisation by mathematicians they are called **hyperbolic functions**, and from certain useful analogies between the geometry of the hyperbola and that of the circle, the names $\sinh x$ and $\cosh x$ are applied to them.§ In § 96 we have already found the x-gradient of e^x. Using it, we obtain

$$\frac{d \sinh x}{dx} = \frac{d}{dx}\{\tfrac{1}{2}(e^x + e^{-x})\} = \tfrac{1}{2}(e^x - e^{-x}) = \cosh x$$

and

$$\frac{d \cosh x}{dx} = \tfrac{1}{2}(e^x + e^{-x}) = \sinh x.$$

* See Classified List, III. A. 8.
† See Classified List, III. A. 10.
‡ See Classified List, III. A. 9.
§ See Classified List, V.

Written as for integration these results are

$$\int \cosh x \, dx = \sinh x$$

and

$$\int \sinh x \, dx = \cosh x \, .$$

The integrals of other hyperbolic functions are equally easy to find, and are tabulated in Section V. of the *Classified Reference List* at the end of this book.

CHAPTER VI.

TRANSFORMATIONS AND REDUCTIONS.

122. Change of Derivative Variable.—From the fundamental idea of X′, the x-gradient of the function X, as explained in § 35, we have in equations between increments or between integrals

$$dX = X' dx$$

or

$$dx = \frac{dX}{X'} \, .$$

Now, it may be easier to find $\int \frac{X}{X'} dX$ than to find $\int X dx$, and it amounts to the same thing. The former integration may be easier if X′ be capable of convenient expression in terms of X. This transformation has been frequently employed already, as, for instance, in §§ 119, 118, 116, etc., etc.

More generally, if $f(X)$ be any function of X when X itself is a function of x, it may facilitate the integration to substitute

$$\int \frac{f(X)}{X'} dX \text{ for } \int f(X) dx \, ,$$

provided either that X′ cancels out of the expression $\frac{f(X)}{X'}$ or else that X′ is expressible in terms of X.

123. Substitution to clear of Roots.—A selection of such transformations is given in Section II. G of the Classified List, under the title of *Substitutions*, page 169.

Thus in II. G. 4 we get rid of roots by taking

$$X=(a+bx)^{\frac{1}{p}} \therefore x=\frac{X^p-a}{b} \therefore x^m=\frac{1}{b^m}(X^p-a)^m$$

and

$$\frac{1}{X'}=\frac{dx}{dX}=\frac{p}{b}X^{p-1}.$$

Therefore

$$\int\frac{x^m dx}{(a+bx)^{q/p}}=\frac{p}{b^{m+1}}\int\frac{(X^p-a)^m}{X^q}\cdot X^{p-1}dX$$

$$=\frac{p}{b^{m+1}}\int(X^p-a)^m X^{p-q-1}dX;$$

a form which is dealt with later on in § 125; but which can be integrated directly if m be an integer by expanding $(X^p-a)^m$ by the binomial theorem.

124. Quadratic Substitution.—Again, in II. G. 5, the function $f(ax^2+bx+c)$ is transformed so as to get rid of the second term involving the first power of the variable. The proper substitution may be arrived at thus: put

$$X=x+\xi \qquad X'=1 \qquad x=X-\xi.$$

Then

$$ax^2+bx+c=aX^2-2a\xi X+a\xi^2+bX-b\xi+c.$$

The two terms in the first power of X cancel if ξ be taken $\frac{b}{2a}$, and then the remaining three terms not involving X become

$$a\xi^2-b\xi+c=\frac{b^2}{4a}-\frac{b^2}{2a}+c=\frac{4ac-b^2}{4a}$$

$$=\text{say } ak \text{ where } k=\frac{4ac-b^2}{4a^2}.$$

We have then the transformation

$$\int f(ax^2+bx+c)dx=a\int f(X^2+k)dX$$

$$\text{with } X=x+\frac{b}{2a} \text{ and } k=\frac{4ac-b^2}{4a^2}.$$

This transformation is used largely in dealing with quadratic surds.*

125. Algebraico-Trigonometric Substitution.—If in this last

* See Classified List, III. B. 13, 14, 15, and 18.

expression k be positive so that its square root can be extracted, *i.e.*, if $4ac > b$ with a and c both positive; then the above integration of, say, (x^2+k) may be thrown into a trigonometrical form by help of the substitution

$$X = \tan^{-1}\frac{x}{k^{\frac{1}{2}}}, \text{ or } x = k^{\frac{1}{2}}\tan X \text{ and } \frac{1}{X'} = \frac{k^{\frac{1}{2}}}{\cos^2 X}.$$

In these terms

$$x^2 + k = k\tan^2 X + k = \frac{k}{\cos^2 X}$$

and therefore

$$\int f(x^2+k)dx = k^{\frac{1}{2}}\int\frac{1}{\cos^2 X}f\left(\frac{k}{\cos^2 X}\right)dX.*$$

Other similar conversions of algebraic into trigonometrical integrations are detailed in the II. G. Section of the Classified List.

126. Interchange of Two Functions.—In § 87 was established the transformation

$$\int X\Xi' dx = X\Xi - \int X'\Xi dx = X\Xi - \int \Xi dX$$

a special simple case of which, already stated in § 80, is $\Xi' = 1$ and $\Xi = x$, or

$$\int X dx = Xx - \int X' x dx = Xx - \int x dX.$$

This general formula may be useful when the function to be integrated, viz. $(X\Xi')$, is not as a whole directly integrable, but is, however, capable of being split into two factors, one of which (Ξ') has its integral (Ξ) directly recognisable.

127. Interchange of any number of Functions.—The operation may be extended to the integration of the product of any number of functions of x according to the result of § 88; but with the multiplication of the number of functions to be dealt with, there is an increase in the complexity of the conditions under which the formula may be useful, and, therefore, a decrease of the probability or frequency of such usefulness.

Transformations, according to this rule, are called **Integration by Parts.**

128. General Reduction in terms of Second Differential Coefficient.—If $f(X)$ be any function of X, and $f'(X)$ its X-gradient; then, X′ being the x-gradient of X,

$$\frac{df(X)}{dx} = X'f'(X) \text{ by § 84}.$$

* See Classified List, II. G. 7.

Also

$$\frac{d\frac{1}{X'}}{dx} = -\frac{1}{(X')^2}\frac{dX'}{dx}.$$

Here $\frac{dX'}{dx}$ is the x-gradient of X′, and is called the "**second differential coefficient of X with respect to** x," or, more simply, the "**second** x**-gradient of** X." It is concisely written X″. Using this notation (X″) and applying § 126 we have

$$\int f'(X)dx = \frac{f(X)}{X'} + \int \frac{f(X)}{(X')^2}.X''dx.$$

Here the given function is $f'(X)$, and the supposition is that it is directly integrable with respect to X, but not so with respect to x. On this supposition the transformation will be of use if it is found that $\frac{f(X)}{(X')^2}.X''$ is directly, or more easily, integrable with respect to x.

129. General Reduction for X^r.—If $f'(X) = X^r$, then $f(X) = \frac{X^{r+1}}{r+1}$; so that in this case the above formula would be

$$\int X^r dx = \frac{X^{r+1}}{(r+1)X'} + \frac{1}{r+1}\int X^{r+1}.\frac{X''}{(X')^2}.dx.$$

In some cases this form may be preferable to $\int \frac{X^r}{X'}.dX$, which would be given by § 122.

130. General Reduction of $x^m X^r$.—If in § 126 one of the two functions whose product is to be integrated be x^m and the other X^r, where m and r are any constant indices, the transformation gives

$$\int x^m X^r dx = \frac{x^{m+1}X^r}{m+1} - \frac{r}{m+1}\int x^{m+1}X^{r-1}X'dx.$$

If this latter quantity be not directly integrable, it may still be capable of being further reduced by the application of other formulas of transformation already explained, so as to finally reduce it to a directly integrable form.

Such a formula is the base of certain **Formulas of Reduction.**

131. Conditions of Utility of Same.—The last formula given is capable of **repeated** application, provided that X′ is proportional either to some power of x or to some power of X, the right-hand integral then reducing to the same general form as the left-hand one. In either case, or again in the case $xX' = a + bX$, it is not difficult to prove that X must be of the form

$$X = a + bx^n.$$

If r be a positive integer, then X^r can be expanded into a finite series of powers of x, which when multiplied by x^m will give another series of powers of x, each term of which can be integrated separately; so that in this case no need of the above reduction formula will arise; although in some cases its use may shorten the work involved. But the formula is useful for repeated reductions if r is negative or fractional.

Various cases of such uses are given in Section IX. of the *Classified Reference List* at the end.

132. Reduction of $x^m(a+bx^n)^r$.

$$\text{If } X = a + bx^n, \text{ then}$$

$$X' = nbx^{n-1} = \frac{n}{x}(X - a)$$

and § 130 gives

$$\int x^m X^r dx = \frac{x^{m+1}X^r}{m+1} - \frac{rn}{m+1}\int x^m X^{r-1}(X-a)dx$$

$$= \frac{x^{m+1}X^r}{m+1} - \frac{rn}{m+1}\int x^m X^r dx + \frac{rna}{m+1}\int x^m X^{r-1} dx.$$

Here we have $\int x^m X^r dx$ on each side. Bringing these two terms to one side, and dividing out by the sum of their numerical factors, viz. $\left(1 + \frac{rn}{m+1}\right) = \frac{m+1+rn}{m+1}$; we find

$$\int x^m X^r dx = \frac{x^{m+1}X^r}{m+1+rn} + \frac{rna}{m+1+rn}\int x^m X^{r-1} dx;*$$

a formula of reduction by which in the integration the power of X is reduced by 1, while that of x is left unchanged. The reduction of the power of X is compensated for by the multiplication (outside the sign of integration) by the factor a, which has the same "dimensions" as X.

This formula can be used inversely to pass from $x^m X^{r-1}$ to $x^m X^r$, that is, to increase the power of X by 1 without changing that of x.

* See Classified List, IX. A. vi.

If the other form of X′, namely, nbx^{n-1}, be used in this transformation, there results

$$\int x^m X^r dx = \frac{x^{m+1}X^r}{m+1} - \frac{rnb}{m+1}\int x^{m+n}X^{r-1}dx\,;$$

a formula of reduction by which, while the power of X is decreased by 1, that of x is increased by n.

By the previous formula $\int x^{m+n}X^{r-1}dx$ may be converted into a quantity in terms of $\int x^{m+n}X^r dx$, and thus $\int x^m X^r$ reduced to an integral in which the power of x is raised by n, while that of X is left unaltered.

By similar transformations one can ring the changes among the integrals of the following set of nine functions, any one of which can be reduced to any other.

$x^m X^{r+1}$	$x^{m+n}X^{r+1}$	$x^{m-n}X^{r+1}$
$x^m X^r$	$x^{m+n}X^r$	$x^{m-n}X^r$
$x^m X^{r-1}$	$x^{m+n}X^{r-1}$	$x^{m-n}X^{r-1}$

The complete set of reduction formulæ for this purpose are given in Section IX. of the *Classified Reference List* at the end.

133. Reduction of r^{th} Power of Series of any Powers of x.—Similarly if

$$X = ax^\alpha + bx^\beta + gx^\gamma + k\,;$$

then § 130 gives

$$\int x^m X^r dx = \frac{x^{m+1}X^r}{m+1} - \frac{r}{m+1}\int x^m(\alpha ax^\alpha + \beta bx^\beta + \gamma gx^\gamma)X^{r-1}dx\,.$$

The last integral may be taken in terms each of the form $\int x^p X^{r-1}dx$, and on account of the reduction from r to $(r-1)$ in the index of X, these may be more amenable to simple integration than the original $\int x^m X^r dx$.

Evidently this formula applies to the sum of any number of terms in different powers of x.

134. Special Case.—If in the last article one index = 0 and another = 1, we have

$$X = a + bx^n + cx$$

in which case a simple reduction, like that of § 132, will show that

$$\int x^m X^r dx = \frac{x^{m+1}X^r}{m+1+nr} + \frac{r}{m+1+nr}\int x^m\{(n-1)cx+na\}X^{r-1}dx\,.$$

135. Trigonometrical Reductions.—If in the general formula of § 126 the product $X\Xi'$ be equal to $\sin^n x$, then we may split this into the two factors $\sin x$ and $\sin^{n-1}x$, thus:—

$$X = \sin^{n-1}x \qquad X' = (n-1)\sin^{n-2}x\cos x$$
$$\Xi' = \sin x \qquad \Xi = -\cos x$$
$$X'\Xi = -(n-1)\sin^{n-2}x\cos^2 x = -(n-1)\sin^{n-2}x(1-\sin^2 x)$$
$$= (n-1)\sin^n x - (n-1)\sin^{n-2}x\,.$$

Therefore

$$\int \sin^n x dx = -\sin^{n-1}x\cos x - (n-1)\int \sin^n x dx + (n-1\int \sin^{n-2}dx$$

$$= -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}dx\ *$$

by adding together on the left-hand side the two terms in $\int \sin^n x dx$ and dividing by the sum of their coefficients, namely, $1+(n-1)=n$.

136. Trigonometrical Reductions.—Since by § 78 the angle-gradient of the tangent is the reciprocal of the square of the cosine, and since $dx = \frac{dX}{X'}$, we have

$$dx = \cos^2 x d(\tan x) = (1-\sin^2 x)d(\tan x)$$

and, therefore,

$$\int \tan^{n-2}x dx = \int \tan^{n-2}x(1-\sin^2 x)d(\tan x) = \frac{\tan^{n-1}x}{n-1} - \int \tan^n x dx\,.$$

Or, rearranging,

$$\int \tan^n x dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x dx\,.$$

By either the process adopted in this article or that of last article, the various trigonometrical formulæ of reduction are established which are set forth in order in Section IX. B. 1 to 9 of the *Classified Reference List* at the end of this book. In each case the reduction changes the index by 2, which change results from the substitution $\sin^2 x = 1 - \cos^2 x$.

* See Classified List, IX. B. 1.

If n the index to be reduced be an even integer, a repetition of the reduction will finally bring down the integral to the form $\int dx$, whose integral is x. If n be odd, the finally reduced integral will be either $\int \sin x dx$ or $\int \cos x dx$ or $\int \tan x dx$, etc., etc., all of which have already been demonstrated.

137. Trigonometrico-Algebraic Substitution. — If in these trigonometrical reductions n is fractional, then since $dx = \frac{dX}{X'}$ and $\frac{d \sin x}{dx} = \cos x$, etc., etc., we may convert the formula into that of § 132 and IX. A. 1 of the *Classified List*, thus:—

$$\int \sin^n x dx = \int \frac{\sin^n x}{\cos x} d(\sin x) = \int \frac{\sin^n x}{(1 - \sin^2 x)^{\frac{1}{2}}} d(\sin x)$$

$$= \int X^n (1 - X^2)^{-\frac{1}{2}} dX$$

in which X stands for $\sin x$.

Similarly

$$\int \sin^n x \cos^m x dx = \int \frac{\sin^n x (1 - \sin^2 x)^{\frac{m}{2}}}{(1 - \sin^2 x)^{\frac{1}{2}}} . d(\sin x)$$

$$= \int X^n (1 - X^2)^{\frac{m-1}{2}} dX .$$

138. Composite Trigonometrical Reduction.—From the elementary formulæ for the sine and cosine of the sum of two angles, in terms of the sines and cosines of these angles, we may write: Since $p = (p - 1) + 1$

$$\sin px = \sin (p - 1) x . \cos x + \cos (p - 1) x \sin x.$$
$$\cos px = \cos (p - 1) x . \cos x - \sin (p - 1) x . \sin x .$$

Therefore integrals of $(\sin^{\pm n} x . \sin px)$, and $(\sin^{\pm n} x . \cos px)$, and $(\cos^{\pm n} x . \sin px)$ and $(\cos^{\pm n} x . \cos px)$ may, by repeated application of the above formulæ, be reduced to series of integrals of the form $(\sin^{\pm n} x \cos^{m \pm} x)$, provided p be an integer. For these latter forms, see *Classified Reference List*, IX. B. 5–8.

CHAPTER VII.

SUCCESSIVE DIFFERENTIATION AND MULTIPLE INTEGRATION.

139. The Second *x*-Gradient.—In fig. 6 we have the area underneath the curve measured between two vertical ordinates called X, and the *x*-gradient of X, or X′, is the vertical ordinate to the curve. The slope of this curve at any point is thus the *x*-gradient of X′, or the *x*-gradient of the *x*-gradient of X. This is called the "**Second Differential Coefficient of X with respect to *x*,**" or the "**Second *x*-gradient of X.**" It is written either

$$\frac{d^2X}{dx^2} \text{ or } X''.$$

In fig. 5 the vertical ordinate is X, and the gradient of the curve is therefore X′. Therefore, in this graphic representation the second *x*-gradient of X is the rate at which the gradient of the curve changes with advance along the *x*-axis. If the gradient of the curve increases with *x*, the diagram line curves upwards and X″ is positive; if the gradient decreases in the same direction there is downward curvature, and X″ is negative. The possible variations of sign of X″ are shown on fig. 3. If there be no curvature, *i.e.*, if the diagram line be straight, then X″ is zero. Although X″ is greater the sharper the curvature, still X″ is not the measure of the curvature. The relation between X″ and the curvature is explained in § 145 below.

140. Increment of Gradient.—In fig. 5 if the gradient increase in the horizontal length δx by $(X'_2 - X'_1) = \delta X'$, then this divided by δx is the average rate of change of X′ throughout the length δx. If the two points be close together, then this is the actual rate of change of X′ between these two contiguous points; that is, it is the value of X″ at this part of the curve. That is, when δx is small, $\frac{\delta X'}{\delta x} = X''$ or $\delta X' = X''\delta x$.

141. Second Increment.—If δx be small, then from the point $(x_1 - \frac{1}{2}\delta x)$ to the point $(x_1 + \frac{1}{2}\delta x)$, the curve of fig. 5 rises a height $X'_1\delta x$, and from the point $(x_1 + \frac{1}{2}\delta x) \equiv (x_2 - \frac{1}{2}\delta x)$ to the point $(x_2 + \frac{1}{2}\delta x)$, the rise of the curve is

$$X'_2\delta x = (X'_1 + \delta X')\delta x = (X'_1 + X''\delta x)\delta x.$$

Thus while the rise is $X'_1\delta x$ through the small stretch δx lying equally on either side of the point 1, it is greater than this by $X''(\delta x)^2$ through the next equal small stretch δx lying equally on either side of the point 2. The horizontal distance between these two points is δx. Thus an advance of δx has resulted in an increase of rise per δx equal to $X''(\delta x)^2$. The excess of the second of these rises over the first is called the **second increment** or **second difference** of X. It might be written $\delta(\delta X)$, but the usual and neater symbol is $\delta^2 X$. Dividing by $(\delta x)^2$ we have

$$\frac{\delta^2 X}{(\delta x)^2} = X'' = \frac{\delta X'}{\delta x}.$$

Stated in words this reads:—

The second x-gradient of X equals the second increment of X per δx per δx divided by the square of δx.

It must be remembered that this equation has been derived only on the supposition that δx is taken very small. The other written symbol for X'', namely, $\frac{d^2 X}{dx^2}$ resembles in form $\frac{\delta^2 X}{(\delta x)^2}$ with d substituted for δ. It is sometimes stated that dx is the symbol for δx when δx becomes "infinitely small," and that when δx becomes "infinitely small" $\frac{\delta^2 X}{(\delta x)^2}$ equals $\frac{d^2 X}{dx^2}$; but the present writer prefers to avoid the use of phraseology which suggests reference to quantities which do not exist, and reasoning about which must, therefore, be quite unmeaning.

It should be noted that $\delta^2 X$ is of the same kind and dimensions as X, and that δx is of the same kind and dimensions as x; so that X'' is of the kind and dimensions of X/x^2.

142. Integration of Second Increment.—From the equation

$$\delta^2 X = X''(\delta x)^2$$

we obtain the other

$$\text{Integral of } \delta^2 X = \text{Integral of a corresponding continuous series of consecutive values of } \{X''(\delta x)^2\}.$$

Remembering the meaning of $\delta^2 X$, it is evident that its integral from one definite point 1 or x_1 to any other point x equals

$$\delta X \text{ at } x - \delta X \text{ at } x_1$$
$$= \text{say } \delta X - (\delta X)_1$$

these two increments being taken for equal small δx's. Both these increments are very small; their difference is minutely small, and, therefore, this integration as it stands can be of no practical use as a final result. But let us integrate again this minutely small difference of increments by taking all the successive values of δX from x_1 up to point x and subtracting the constant $(\delta X)_1$ from each, and then let us add all the differences. The result is (by the distributive law in integration, see § 83) the same as $\int_{x_1}^{x}\delta X - \int_{x_1}^{x}(\delta X)_1$. In the latter symbol, since $(\delta X)_1$ has a constant value, we have simply to multiply this by the number of them to be added, which number is $\frac{x - x_1}{\delta x}$; so that

$$\int_{x_1}^{x}(\delta X)_1 = \frac{x - x_1}{\delta x}\,.\,(\delta X)_1 = (x - x_1)\left(\frac{\delta X}{\delta x}\right)_1 = (x - x_1)X'_1\,.$$

In fig. 5 X'_1 is the gradient at the point 1, and this last, therefore, equals the rise of the tangent at the point 1 in the horizontal stretch $(x - x_1)$. Also $\int_{x_1}^{x}\delta X$ is the whole rise of the curve through this same stretch, or $(X - X_1)$ Therefore, this double integration of $\delta^2 X$, the shorthand symbol for which is $\int\int d^2X$, gives the result

$$\int_{x_1}^{x}\int_{x_1}^{x} d^2X = (X - X_1) - (x - x_1)X'_1$$
$$= X - xX'_1 - (X_1 - x_1X'_1)$$

and this is evidently measured in its graphic representation (see fig. 26) by the height of the curve above its tangent at point 1.

143. Graphic Delineation.—In the process of single integration we found one constant introduced. In this double integration there are two introduced. In the above algebraic representation they are $-X'_1$ and $-(X_1 - x_1X'_1)$.

On fig. 26 are clearly marked the graphic representations of these two constants, the first being the gradient of the curve at point 1.

The lower limit is the same in both steps of this double integration, namely, the point 1. It is not necessarily so. In the first step, the lower limit may be the point 1, and this will give the gradient-constant X'_1. In the second integration the lower limit might be taken at some other point 2, and this would give

$$\int_2\int_1 d^2X = (X - X_2) - (x - x_2)X'_1$$
$$= X - xX'_1 - (X_2 - x_2X'_1)$$

leaving the one constant still *minus* the curve-gradient at 1, but making now the second constant *minus* the intercept upon the vertical axis cut off by a line through 2 parallel to the tangent at 1. Otherwise, the integral is now the height of the curve above the line drawn through 2 parallel to the tangent at 1.

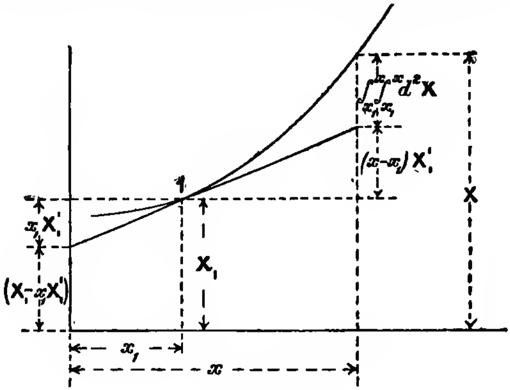

FIG. 26.

144. Integration of Second x-gradient.—Evidently $\int X''dx = X' + C$. Integrating a second time

$$\int \left\{ \int X''dx \right\} dx = \int \left\{ X' + C \right\} dx = X + Cx + K.$$

This formula is identical with the above if

$$C = -X'_1 \text{ and } K = -(X_2 - x_2 X'_1).$$

This double integration of X'' is written shorthand $\int\int X''dx\,dx$, or still more shortly $\int^{II} X''dx^2$.

We thus see that when proper attention is paid to the equality of the constants of integration,

$$\int\int d^2X = \int\int X''dx\,dx$$

or

$$\int^{II} d^2X = \int^{II} X''dx^2.$$

145. Curvature.—To show the relation between X'' and the curvature of the graphic representation, let a be the angle which the curve at any point makes with the axis of x, and let this increase to $(\alpha+\delta\alpha)$ in the arc-length δa, whose horizontal projection is δx. The curvature is the reciprocal of the radius of curvature, or the "arc-gradient of the angle," *i.e.*, $\frac{d\alpha}{da}$. Since $\delta\alpha$ is small, it equals $\tan(\delta\alpha)$, and $\tan(\delta\alpha)$ can be calculated by the ordinary trigonometrical rule from X' and $(X'+\delta X')$ the tangents of α and $(\alpha+\delta\alpha)$; thus,

$$\delta\alpha=\tan(\delta\alpha)=\frac{(X'+\delta X')-X'}{1+(X'+\delta X')X'}=\frac{\delta X'}{1+X'^2} \text{ nearly, when } \delta x \text{ is small.}$$

Also the arc-length is

$$\delta a=\sqrt{\delta x^2+\delta X^2}=\delta x\sqrt{1+X'^2}.$$

Therefore the curvature is, ρ being the radius of curvature,

$$\frac{1}{\rho}=\frac{d\alpha}{da}=\frac{\delta X'}{\delta x}\cdot\frac{1}{\{1+X'^2\}^{\frac{3}{2}}}=\frac{X''}{\{1+X'^2\}^{\frac{3}{2}}}.$$

If the radius of curvature be easily found by any direct process, the inverse form of the above relation may be useful; namely,

$$X''=\frac{1}{\rho}\{1+X'^2\}^{\frac{3}{2}}.$$

If T be the sub tangent on the x-axis, and if (see fig. 27) the

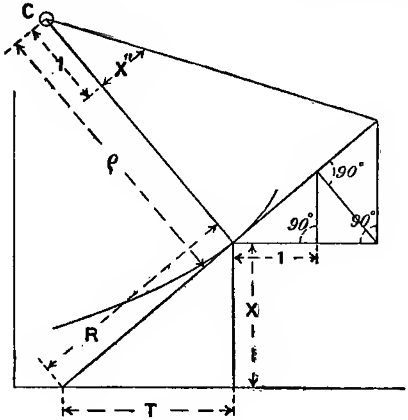

FIG. 27.

intercept on the tangent between this axis and the touching point be called R; then since $R^2=X^2+T^2$ and $X'=\frac{X}{T}$, therefore

$$1+X'^2=\frac{R^2}{T^2}$$

and

$$X''=\frac{1}{\rho}\left(\frac{R}{T}\right)^3.$$

If the radius of curvature be known—(a practised draughtsman can always find it with the greatest accuracy in two or three seconds by one or two trials with the dividers)—the construction shown in fig. 27 affords a very easy graphic method of finding X″, measured to scale, according to the above formula $\frac{1}{\rho}\left(\frac{R}{T}\right)^3$. Those acquainted with the elements of Graphic Calculation will readily follow the construction from the marking of the figure without further explanation.

146. Harmonic Function of Sines and Cosines.—As an illustration of these ideas, take an ordinary harmonic curve. Let h be the height of the curve at horizontal ordinate l; let r_1, r_2 and m be constants; and let

$$h=r_1 \sin ml+r_2 \cos ml$$

then

$$h'=mr_1 \cos ml - mr_2 \sin ml$$

and

$$h''=-m^2(r_1 \sin ml+r_2 \cos ml)=-m^2h.$$

The student should write out these results for the three simplified cases—(1) $r_1=r_2=r$; (2) $r_2=0$; and (3) $r_1=0$. In all cases he will find that $h''=-m^2h$.

147. Deflection of a beam.—If a beam be uniformly loaded with a load w per ft. run and have a vertical supporting force R applied at one end, the bending moment on the section distant l from this end of the beam is

$$M=Rl-\tfrac{1}{2}wl^2.$$

The bending moment diagram (ordinates M and l) is therefore a parabola. At any point l the gradient of the curve is

$$\begin{aligned}M'&=R-wl\\&=R \text{ at the end where R acts}\\&=0 \text{ at section where } wl \text{ equals R}\end{aligned}$$

between which points it varies uniformly.

The second l-gradient of M is

$$M'' = -w$$

and is thus constant.

It is easily shown that, in a beam subjected to elastic bending only, the curvature of the (originally straight) axis equals $\frac{M}{EI}$, where I is the "**area-moment of inertia**" of the section, and E is the modulus of elasticity.

In the case of beams so stiff that the bending under safe loads is very small—which is the only case of practical interest to engineers—it is sufficiently accurate to take the curvature as the second l-gradient of the deflection, neglecting the division by the $\frac{3}{2}$ power of 1 *plus* the square of the first l-gradient.

Thus if Δ be the deflection perpendicular to l, then the second l-gradient of Δ or

$$\Delta'' = \frac{M}{EI} = \frac{1}{EI}(Rl - \tfrac{1}{2}wl^2) \text{ in above case}$$

and

$$\Delta = \int^{II} \Delta'' dl^2 = \frac{1}{EI}\int^{II}(Rl - \tfrac{1}{2}wl^2)dl^2$$ if both E and I are constant along l:

$$= \frac{1}{EI}\int\left(EI\,\Delta'_1 + \frac{R}{2}l^2 - \frac{w}{6}l^3\right)dl$$ if Δ'_1 defines the gradient of the line from which the deflection is to be measured:

$$= \Delta_2 + \Delta'_1 l + \left(\frac{R}{6}l^3 - \frac{w}{24}l^4\right)\frac{1}{EI}$$ where Δ_2 defines the position of the line from which the deflection is measured.

At any point l the gradient is

$$\Delta' = \Delta'_1 + \left(\frac{R}{2}l^2 - \frac{w}{6}l^3\right)\frac{1}{EI}.$$

If the deflection be measured from a line parallel to the bent axis at its point where $wl = R$, then Δ' must equal 0 at this point, and this gives $\Delta'_1 = -\frac{R^3}{3EIw^2}$. If, further, we measure from a line drawn through the ends of the axis so as to make the end deflections zero, then Δ must be zero at $l=0$, which gives $\Delta_2 = 0$.

Inserting these values of the two constants, we find

$$\Delta = \frac{1}{3EI}\left(-\frac{R^3}{w^2}l + \frac{R}{2}l^3 - \frac{w}{8}l^4\right).$$

This equals zero when $l=0$; and, when $wl=\mathrm{R}$, which occurs at the centre of a uniformly loaded beam freely supported at both ends, it equals

$$\Delta_c=\frac{\mathrm{R}^4}{3\mathrm{EI}w^3}\left(-1+\tfrac{1}{2}-\tfrac{1}{8}\right)=-\frac{5}{24}\frac{\mathrm{R}^4}{\mathrm{EI}w^3}.$$

If the span be L, the whole load $=w\mathrm{L}=\mathrm{W}$ and $\mathrm{R}=\frac{1}{2}w\mathrm{L}$, and

$$\Delta_c=-\frac{5}{384}\frac{w\mathrm{L}^4}{\mathrm{EI}}=-\frac{5}{384}\frac{\mathrm{WL}^3}{\mathrm{EI}}.$$

148. Double Integration of Sine and Cosine Function.—In § 146 we find

$$h''=-m^2h$$

and, comparing this with the original equation, we see that the general result of a double integration from this relation is

$$h=r_1\sin ml+r_2\cos ml$$

where r_1 and r_2 are the two constants introduced by integration.

But if the second-gradient equation be of the other form

$$h''=-m^2(r_1\sin ml+r_2\cos ml)$$

the result of a double integration is not the same: it is more general, namely,

$$h=r_1\sin ml+r_2\cos ml+\mathrm{C}_1l+\mathrm{C}_2$$

where C_1 and C_2 are the two integration constants.

The former $(h''=-m^2h)$ is a special case of the latter more general rule, in which special case $\mathrm{C}_1=0=\mathrm{C}_2$; and this specialty gives rise to the relation $h''=-m^2h$, which relation does not hold good in general.

In the general formula the constant C_1 gives a choice of gradient of the line from which h is to be measured; while C_2 gives a further choice of level at which to draw this datum line. In the special case this level must be such as to make $h=r_2$ when $l=0$ and the gradient of the datum line must be zero.

149. Exponential Function.—If $\mathrm{X}=b^x$, then by § 95

$$\mathrm{X}'=\frac{b^x}{\mathrm{T}}=\frac{\mathrm{X}}{\mathrm{T}}\text{ and therefore }\mathrm{X}''=\frac{\mathrm{X}}{\mathrm{T}^2}.$$

If $\mathrm{X}=b^{mx}$ where m is any constant, either positive or negative, whole or fractional; then

$$\mathrm{X}'=\frac{m}{\mathrm{T}}\mathrm{X}\text{ and }\mathrm{X}''=\left(\frac{m}{\mathrm{T}}\right)^2\mathrm{X}.$$

This case is the counterpart of that in the last article where $(-m^2)$ was essentially and necessarily negative, while here $\left(\frac{m}{\mathrm{T}}\right)^2$ is essentially positive.

150. Product and Quotient of two or more x-Functions.—If L and M are functions of x, and if $\mathrm{X}=\mathrm{LM}$, the first and second x-gradients of X are

$$\mathrm{X}' = \mathrm{L}'\mathrm{M} + \mathrm{L}\mathrm{M}' \text{ and}$$

$$\therefore \mathrm{X}'' = \mathrm{L}''\mathrm{M} + 2\mathrm{L}'\mathrm{M}' + \mathrm{L}\mathrm{M}''.$$

In the case of $\mathrm{M}=x$, then $\mathrm{M}'=1$ and $\mathrm{M}''=0$;

$$\text{therefore } \mathrm{X}'' = \mathrm{L}''x + 2\mathrm{L}'.$$

Similarly, if X be the product of three x-functions, L, M, N, then X″ is the sum of a series of terms each of which contains the three letters L M and N, and in each term the number of dashes indicating the number of differentiations will be 2. Dividing by $\mathrm{X}=\mathrm{LMN}$, the result may be written

$$\frac{\mathrm{X}''}{\mathrm{X}} = \frac{\mathrm{L}''}{\mathrm{L}} + \frac{\mathrm{M}''}{\mathrm{M}} + \frac{\mathrm{N}''}{\mathrm{N}} + 2\left(\frac{\mathrm{L}'\mathrm{M}'}{\mathrm{LM}} + \frac{\mathrm{L}'\mathrm{N}'}{\mathrm{LN}} + \frac{\mathrm{M}'\mathrm{N}'}{\mathrm{MN}}\right)$$

a form analogous to that of § 92; and which may be extended to the product of any number of factors.

151. Third and Lower x-Gradients and Increments.—The x-gradient of the second x-gradient of X is the third x-gradient, and the x-gradient of this again is the fourth x-gradient; and so on through any number of differentiations.

These successive gradients are written either X''', X^{iv}, X^{v}, or else $\frac{d^3\mathrm{X}}{dx^3}$, etc.

Similarly, if two successive values of the second increment of X per δx per δx be taken at two places δx apart, *the difference between them* is called the third increment of X per δx per δx per δx. This is written $\delta^3\mathrm{X}$; and if it be divided by the cube of δx, it is easy to show that $\frac{\delta^3\mathrm{X}}{(\delta x)^3} = \mathrm{X}''' = \frac{d^3\mathrm{X}}{dx^3}$ when δx is very small. This is **not** a truism. The symbol $\frac{d^3\mathrm{X}}{dx^3}$ ought **not** to be considered capable of being split into two parts, one of which, the numerator, $d^3\mathrm{X}$, is the value of $\delta^3\mathrm{X}$ when δx is very small. Nevertheless, it is correct to write for **any** very small δx

$$\delta^3\mathrm{X} = \mathrm{X}'''(\delta x)^3 = \frac{d^3\mathrm{X}}{dx^3}(\delta x)^3.$$

Again, if $\delta^n X$ be the n^{th} increment, and $\frac{d^n X}{dx^n}$ the n^{th} gradient, then for any very small δx it is correct to write

$$\delta^n X = \frac{d^n X}{dx^n} (\delta x)^n .$$

152. Rational Integral x-Functions.—If $X = kx^m$, then

$$X' = kmx^{m-1} ; \; X'' = km(m-1)x^{m-2}$$

and

$$\frac{d^n X}{dx^n} = km(m-1)(m-2)\text{----}(m+1-n)x^{m-n} .$$

If m be a positive integer, the m^{th} gradient of kx^m will be a constant; namely,

$$\frac{d^m X}{dx^m} = km(m-1)(m-2)\text{----}3 \cdot 2 \cdot 1 = km!$$

Thus the $(m+1)^{\text{th}}$ gradient of kx^m is zero, as is also every lower gradient, if m be a positive integer.

But if m be fractional, then the successive gradients pass into negative powers of x, so that, in this case, a lower gradient may have a very large value for very small values of x. Thus, if $X = x^{\frac{5}{2}}$; then, $X' = \frac{5}{2}x^{\frac{3}{2}}$, $X'' = \frac{15}{4}x^{\frac{1}{2}}$, and $X''' = \frac{15}{8x^{\frac{1}{2}}}$; giving very large values of X''' for very small values of x.

If $X = ax^m + bx^{m-1} + \text{----} + kx$, and if m be an integer, then at each successive differentiation one term disappears, and the m^{th} gradient is again a constant, viz., $am!$ Thus any terms in the above function, except the first, might be omitted without altering the m^{th} gradient. There are, therefore, $(m-1)!$ different functions of the above type which give the same m^{th} gradient; $(m-2)!$ different ones which give the $(m-1)^{\text{th}}$ gradient the same in all; and so forth, the differences corresponding with those arising from putting any except the first of the constant factors in the above general formula equal to zero.

153. Lower x-Gradient of Sine and Exponential Functions.—The successive gradients of some functions have a re-entrant or repeating character. For instance,

$$X = k_1 \sin mx + k_2 \cos mx$$

$$X'' = -m^2 X$$

$$\therefore X^{\text{iv}} = m^4 X$$

$$\therefore X^{\text{vi}} = -m^6 X, \text{ etc., etc.}$$

Again, see § 95,

$$X = b^{\beta x}$$

$$X' = \frac{\beta}{T} b^{\beta x} = \frac{\beta}{T} X$$

$$\therefore X'' = \left(\frac{\beta}{T}\right)^2 X$$

$$\therefore \frac{d^n X}{dx^n} = \left(\frac{\beta}{T}\right)^n X$$

and this is true whether β be + or −.*

154. General Multiple Integration.—If X', X'', X''', X^{iv}, etc., are the successive x-gradients of some function X, and if we start with a knowledge of the lowest of these gradients only, and wish to work upwards to a knowledge of the higher gradients and of X by repeated integration; we find

$$\int X^{iv} dx = X''' + C_3$$

where C_3 is the constant of integration. Then

$$\int\int X^{iv} dx^2 = \int X''' dx + \int C_3 dx = X'' + C_3 x + C_2$$

and

$$\int^{III} X^{iv} dx^3 = X' + \frac{C_3}{2} x^2 + C_2 x + C_1$$

and

$$\int^{IV} X^{iv} dx^4 = X + \frac{C_3}{6} x^3 + \frac{C_2}{2} x^2 + C_1 x + C_0 .$$

This result might perhaps be more clearly understood when expressed as follows:—X^{iv} may be written $(X^{iv} + 0)$. Then the proposition is that the fourth integral of the known function X^{iv} is the function X whose fourth x-gradient is X^{iv}, *plus* the function $(\frac{1}{6}C_3 x^3 + \frac{1}{2}C_2 x^2 + C_1 x + C_0)$ whose fourth x-gradient is 0.

If there were n integrations, there would be $(n+1)$ terms in the result, one of which would be a constant, and $(n-1)$ of which would be multiples of the first $(n-1)$ integral powers of x. § 152 illustrates one special example of this general proposition.

The constants are to be determined from the "**limiting conditions.**" The number of limiting conditions, a knowledge of which is necessary to definitely solve the problem, is the same as the number of "**arbitrary constants**" C appearing in the general solution.

* See Appendix F.

In the above case C_2 might be determined from a knowledge of one particular value of X'', and then C_1 from that of one particular gradient X', the remaining C_0 being found from one particular value of X being given.

155. General Multiple Integration.—If in § 126 we write $\int Xdx$ instead of X, and therefore X instead of X′, we obtain

$$\int\{\Xi'\int Xdx\}dx = \Xi\int Xdx - \int \Xi X dx.$$

If in this formula the x-function Ξ be x itself, so that $\Xi' = 1$, there results

$$\int^{II} Xdx^2 = x\int Xdx - \int xXdx$$

which enables two possibly easy single integrations to be substituted for one double integration which may be otherwise impracticably difficult.

Conversely, a given function (xX'') may be difficult to integrate once, while the part of it X'' is recognised as the second x-gradient of a known function X, and then the form

$$\int xX''dx = xX' - \int^{II} X''dx^2$$
$$= xX' - X$$

may be useful.

156. Reduction Formulæ.—From § 150 we have

$$\frac{d^2}{dx^2}(xX) = xX'' + 2X'$$

from which it follows that

$$xX = \int^{II} xX''dx^2 + 2\int^{II} X'dx^2.$$

In this substitute X for X′, and therefore X′ for X″, and $\int Xdx$ for X ; there results then

$$\int^{II} Xdx^2 = \tfrac{1}{2}x\int Xdx - \tfrac{1}{2}\int^{II} xX'dx^2$$
$$= \tfrac{1}{2}x\int Xdx - \tfrac{1}{2}\int^{II} x\,dX\,dx.$$

Again, if in the same formula there be substituted X for X″, and therefore $\int X dx$ for X′ and $\int^{\text{II}} X dx^2$ for X, the result appears as

$$\int^{\text{III}} X dx^3 = \tfrac{1}{2} x \int^{\text{II}} X dx^2 - \tfrac{1}{2} \int^{\text{II}} x X dx^2 .*$$

157. Graphic Diagram of Double Integration.—The meaning of double integration can be very easily represented graphically.

In fig. 5 the slope of the curve is X′ and the height of the curve is X, the first integral of X′ by dx. Thus $(X\delta x)$ or the strip of area between two contiguous verticals under the curve is the increment of the second integral of X′ by dx. Thus the area under the curve included between two given limiting verticals is their second integral, or

$$\text{Area under curve} = \int^{\text{II}} X' dx^2.$$

This graphic representation will help the student to perceive clearly that this integral is **not** the sum of a number of terms, each of which is the square of δx multiplied by the slope X′. The square of any one δx multiplied by the coincident slope X′ would be the rectangle of base δx and height δX, because $X'\delta x = \delta X$. The sum of the series of such rectangular areas stretching between given limits on the curve is not any definite area, and it can be made as small as desired by taking the δx's sufficiently small. But this small rectangular area $(\delta x . \delta X)$ is easily recognised to be the *second increment* of the area under the curve. The first difference is the area of the whole vertical strip between contiguous verticals. The difference between two such successive narrow strips (each being taken the same width δx) is the above $(\delta x . \delta X)$. Thus as $X' dx^2$ is this second difference which equals $X'(\delta x)^2$, there is nothing illegitimate in considering the symbol dx^2 in $\int^{\text{II}} X' dx^2$ to represent the value of $(\delta x)^2$ when δx is taken minutely small.

158. Graphic Diagram of Treble Integration.—The idea of treble integration may be similarly represented graphically.

If the various areas in fig. 5 under the curve measured from any given lower limit up to the various vertical ordinates at the successive values of x, be looked upon as projections or plan-sections of a solid, the successive sections for each x and the following $(x + \delta x)$ being raised above the paper by the heights x and $(x + \delta x)$; then this volume is the true graphic representation of $\int^{\text{III}} X' dx^3$, because the increment of this volume, or the slice of volume lying between

* See Appendix G.

two successive parallel sections δx apart, is the section-area at the middle of the thickness δx multiplied by δx. This section-area we have seen in § 157 to be $\int^{\text{II}} X'dx^2$; and, therefore, the above increment of volume is $\int^{\text{II}} X'dx^3$. The integral of this is $\int^{\text{III}} X'dx^3$.

If the lower limiting vertical ordinate of the area be at $x=0$, then two of the side surfaces of the above integral volume are **planes** normal to the paper of the diagram and passing through the axes of x and X. A third side surface, namely, that passing through the successive X edges (which are the various upper limiting ordinates in the area integrals), is also a **plane**: it passes through the X-axis and is inclined at 45° to the diagram paper. The fourth side surface is in general curved. These four side surfaces, three of which are flat, give to the volumetric representation of treble integration the general form of a quadrilateral pyramid. The base of this pyramid is plane parallel to the diagram paper.

As a valuable exercise, the student should endeavour to obtain a clear mental conception of the fact that $X'(\delta x)^3$, the value of which becomes $X'dx^3$ when δx is minutely small, is the third difference in the continuous increase with x of this pyramidal volume.

CHAPTER VIII.

INDEPENDENT VARIABLES.

159. Geometrical Illustration of Two Independent Variables.—Hitherto there have been considered combinations of such functions alone as are mutually dependent on each other. The functions x, X, $\mathfrak{X}$, etc., have been such that no one of them can change in size without the others concurrently changing size.

In fig. 1, § 11, we have a vertical plane section of the surface of a piece of undulating land. Suppose it to be a meridional or north and south section. On it each distance measured northwards from a given starting-point corresponds to a definite elevation of the ground. If we take other meridional sections of the same piece of country, this same northward co-ordinate will correspond with other heights in these other sections. Thus, if h be used as a general symbol to mean the height of the surface at any and every point of it, then h depends not only on the northward co-ordinate or latitude, but also upon the westward co-ordinate or longitude. If there be freedom to move anywhere over the surface, the two co-ordinates of latitude and longitude may be varied **independently of each other**, that is, a change in one does not necessitate any change in the other.

Under such circumstances the **elevation** is said to be a **function of two independent variables.**

160. Equation between Independent Increments.—In moving from any point 1 to any other point 2, the elevation rises (or falls) from say h_1 to h_2. Let the latitudes, or northward ordinates, of the two points be n_1 and n_2, an l the westward ordinates or longitudes be w_1 and w_2. Then the same change of elevation would be effected by either of two pairs of motions; namely, first, a motion northwards $(n_2 - n_1)$ without change of longitude, followed by a motion $(w_2 - w_1)$ westwards without change of latitude; or, second, a motion $(w_2 - w_1)$ without change of n, followed by a motion $(n_2 - n_1)$ without change of w. This is true whether these motions be large or small. Suppose them to be small, and further suppose that there are no sudden breaks in the ground, that is, that the change of elevation is **continuous** or gradual over the whole surface. Call the small northward, westward, and vertical movements by the symbols

$$n_2 - n_1 = \delta n$$
$$w_2 - w_1 = \delta w$$
$$h_2 - h_1 = \delta h\,.$$

Then if the meridional northward slope of the ground, just north of point 1, be called $\left(\frac{\partial h}{\partial n}\right)_1$, the rise during the small northward movement δn from 1 will be $\left(\frac{\partial h}{\partial n}\right)_1 . \delta n$; and if the westward slope of the parallel of latitude through 2, just east of the point 2, be called $\left(\frac{\partial h}{\partial w}\right)_2$, the rise during the small westward movement δw which, following the above, completes the motion to 2, will be $\left(\frac{\partial h}{\partial w}\right)_2 . \delta w$. The sum of these two rises gives the whole of δh, or

$$\delta h = \left(\frac{\partial h}{\partial n}\right)_1 . \delta n + \left(\frac{\partial h}{\partial w}\right)_2 . \delta w\,.$$

Here the two gradients are not gradients at the same point. If fig. 28 be a plan and two elevations of the small part of the surface considered, they are the northward and westward gradients at ν_1 and ν_2 at the middle points of 1N and N2 in the plan.

If now the passage from 1 to 2 be effected by passing through W in fig. 28, and if $\left(\frac{\partial h}{\partial w}\right)_1$ and $\left(\frac{\partial h}{\partial n}\right)_2$ be the westward and north-

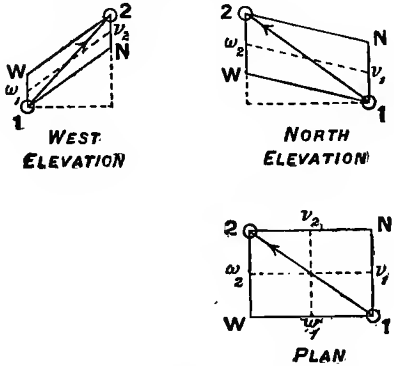

FIG. 28.

ward slopes of the ground at ω_1 and ω_2; then the same change of elevation may be calculated thus,

$$\delta h = \left(\frac{\partial h}{\partial w}\right)_1 \delta w + \left(\frac{\partial h}{\partial n}\right)_2 \delta n$$

where the δw and the δn are also the same lengths as before, the quadrilateral 1N2W being a parallelogram.

$\left(\frac{\partial h}{\partial w}\right)_2$ and $\left(\frac{\partial h}{\partial w}\right)_1$ are the westward slopes on opposite sides of this parallelogram; they are the slopes of N2 and 1W in the "North Elevation." $\left(\frac{\partial h}{\partial n}\right)_1$ and $\left(\frac{\partial h}{\partial n}\right)_2$ are the northward slopes upon the other pair of opposite sides; they are the slopes of 1N and W2 in the "West Elevation."

Adding these two equations and dividing each side by 2; and, further, calling the means between the gradients on the opposite sides of the parallelogram by the symbols $\frac{\partial h}{\partial n}$ and $\frac{\partial h}{\partial w}$; we have

$$\delta h = \frac{\partial h}{\partial n} . \delta n + \frac{\partial h}{\partial w} . \delta w .$$

On a continuous surface such as is here supposed, the above arithmetic means are, with great accuracy, equal to the actual gradients along the centre lines $\omega_1\nu_2$ and $\nu_1\omega_2$ of the small rectangle; that is, the gradients at the centre of the short straight line 1 2.

161. Equation between Independent Gradients.—If the short level length 1 2 be called δs, so that δn and δw are the northward and westward projections or components of δs; then we have, as general truths, by dividing successively by δn, δw, and δs,

$$\frac{\delta h}{\delta n}=\left(\frac{dh}{dn}\right)_s=\frac{\partial h}{\partial n}+\frac{\partial h}{\partial w}\cdot\left(\frac{dw}{dn}\right)_s$$

$$\frac{\delta h}{\delta w}=\left(\frac{dh}{dw}\right)_s=\frac{\partial h}{\partial n}\left(\frac{dn}{dw}\right)_s+\frac{\partial h}{\partial w}$$

$$\frac{\delta h}{\delta s}=\frac{dh}{ds}=\frac{\partial h}{\partial n}\cdot\left(\frac{dn}{ds}\right)_s+\frac{\partial h}{\partial w}\cdot\left(\frac{dw}{ds}\right)_s$$

where the restrictive symbol $(\ )_s$ indicates a ratio of increments occurring concurrently along the special path s over the surface, an element of which path is 1 2 or δs; while the ratios of increments not marked with this symbol are pure northward and eastward gradients. $\frac{dh}{ds}$ does not need to be marked, as its terms indicate plainly that it means the actual whole gradient of the ground along the path s.

$$\left(\frac{dn}{dw}\right)_s,\ \left(\frac{dw}{dn}\right)_s,\ \left(\frac{dn}{ds}\right)_s,\ \text{and}\ \left(\frac{dw}{ds}\right)_s$$

are different measures of the direction of the path s in plan; the first two are the tangents of the inclination of this path from the west and from the north respectively; the last two are the sines of the same inclinations. These measures of its direction particularise the special path to which the equations apply. $\frac{\partial h}{\partial n}$ and $\frac{\partial h}{\partial w}$ have no connection with, and are quite independent of, the direction of this path s: they are the due north and due west gradients at a point of the path, and depend upon the position of this point in the field, but not upon the direction of the path at such point.

The gradients $\frac{\partial h}{\partial n}$ and $\frac{\partial h}{\partial w}$ are called the "**partial**" differential coefficients or gradients of h with respect to n and w.

$\left(\frac{dh}{dn}\right)_s$ is the ratio of rise to northward progress in travelling along the path s, and depends upon the direction of this path. It is quite different from $\frac{\partial h}{\partial n}$. In fig. 28 it, $\left(\frac{dh}{dn}\right)_s$, equals the tangent of inclination of the line 1 2 to the horizontal base in the "West Elevation"; while $\frac{\partial h}{\partial n}$ equals the tangent of inclination of line 1N to same base also in the "West Elevation."

Similarly, $\left(\frac{dh}{dw}\right)_s$ equals the tangent of inclination of 12 to the horizontal base in the "North Elevation," while $\frac{\partial h}{\partial w}$ is the tangent of inclination of line 1W to same base also in the "North Elevation."

162. Constraining Relation between Three Variables.—We have above considered the ordinates n and w to any point of the surface as mutually independent of each other, and h as dependent upon *both* n and w. But we may equally well consider w a function dependent on *both* n and h, while looking on n and h as mutually independent of each other. Generally between the three functions n, w, and h there is only one restrictive relational law established, leaving one degree of freedom of variation among the three. If a second restrictive law be imposed upon the relations between the three, this means that we are restricted to some particular path, such as s, over the surface, and are no longer free to take points all over the surface.

163. Equation of Contours.—The meridional section is such a restricted path; the restriction being $\delta w = 0$. The parallel of latitude is another such restricted path; the restriction in this case being $\delta n = 0$. A level contour line is a third example of such a restricted path, the restriction being $\delta h = 0$. Therefore, if the path s be a contour line, we have $\frac{dh}{ds} = 0$, and thus one form of the equation giving the shape of a contour is

$$\frac{\partial h}{\partial n}\left(\frac{dn}{ds}\right)_s + \frac{\partial h}{\partial w}\left(\frac{dw}{ds}\right)_s = 0$$

or

$$\frac{\left(\frac{dn}{ds}\right)_s}{\left(\frac{dw}{ds}\right)_s} = \left(\frac{dn}{dw}\right)_s = -\frac{\frac{\partial h}{\partial w}}{\frac{\partial h}{\partial n}}.$$

Here $\left(\frac{dn}{dw}\right)_s$ = tangent of northward bearing of contour from due west, and this is seen to equal *minus* the ratio of the due west slope to the due north slope. The *minus* simply means that if both these slopes are positive upward gradients, then the bearing is south, not north, of due west. The steeper the west slope is in comparison with the north slope, the more does the contour veer round to the south.

The geometrical linear ordinates of the above illustration may be taken as the graphic scaled representatives of any kinds of measurable quantities related to each other in a similar manner.

164. General x, y, $F(x,y)$ Nomenclature.—Let the two independent variables be called x and y, and let the function dependent on these be called $F(x,y)$. Let also the rate of change of $F(x,y)$ with x when y is kept constant be called $F'_x(xy)$, and its rate of change with y when x is kept constant be called $F'_y(x,y)$.

Then the equations of § 161 are written

$$\{F'_x(x,y)\} = F'_x(x,y) + F'_y(xy)\frac{dy}{dx}$$

$$\{F'_y(x,y)\} = F'_x(x,y)\frac{dx}{dy} + F'_y(x,y)$$

where $\{F'_x(xy)\}$ is the rate of change of $F(xy)$ with change of x when the change of x is associated with a change of y in the ratio indicated by $\frac{dy}{dx}$; this ratio $\frac{dy}{dx}$ being any whatever, but the ratio inserted on the right hand being always the same as that involved implicitly on the left hand.

165. Two Functions of Two Independent Variables.—Again, if $f(xy)$ be another similar function of x and y, then

$$\left\{\frac{dF(xy)}{df(xy)}\right\} = \frac{F'_x(x,y) + F'_y(x,y)\cdot\frac{dy}{dx}}{f'_x(x,y) + f'_y(x,y)\cdot\frac{dy}{dx}}$$

the brackets { } on the left meaning that the equation gives a particular value of the $f(xy)$ - gradient of $F(xy)$; namely, that particular value obtaining along with change of y combined with change of x in the ratio $\frac{dy}{dx}$ inserted on the right hand.

166. Applications to p, v, t and ϕ Thermal Functions.—An important example of the kind of relation described is that of temperature, pressure, and specific volume of any one definite substance. If t, p, and v indicate these, and if t'_p indicate the pressure-gradient of the temperature with volume kept constant, while t'_v indicates the volume-gradient of the temperature at constant pressure; then for any changes δp and δv of the pressure and volume, there results a temperature increment

$$\delta t = t'_p.\delta p + t'_v.\delta v.$$

For any change of thermal condition in which the volume-gradient of the pressure is $\frac{dp}{dv}$, the volume and pressure gradients of the temperature are

$$\left(\frac{dt}{dv}\right) = t'_p.\frac{dp}{dv} + t'_v$$

and

$$\left(\frac{dt}{dp}\right) = t'_p + t'_v.\frac{dv}{dp}.$$

Or again, if p'_t and p'_v be the temperature and volume gradients of the pressure with volume and temperature respectively kept constant, then for any change defined by a volume-gradient of temperature $\frac{dt}{dv}$, the temperature and volume gradients of the pressure are

$$\left(\frac{dp}{dt}\right) = p'_t + p'_v\frac{dv}{dt}$$

and

$$\left(\frac{dp}{dv}\right) = p'_t.\frac{dt}{dv} + p'_v.$$

Here p'_v is the slope of the "**isothermal**" on the p,v diagram; p'_t is the slope of the "**isometric**" on the p,t diagram; t'_v is the slope of the "**isobaric**" on the t,v diagram, etc., etc.

What is called "**Entropy**," usually symbolised by ϕ, is most simply defined by the equation of its increment

$$\delta\phi = \frac{\delta p.\delta v}{\delta t}$$

and this, combined with the above equations, gives most of the mathematical formulas of thermodynamics.

167. x-**Gradient of** (xy).—In fig. 17, § 80, the rectangular area between the two axes and the two co-ordinates x and X was taken as a function of these co-ordinates, and differentiated with respect to them. The problem was there considered in reference to the ordinates to the particular curve shown in fig. 17, which may be regarded as similar to the particular curve s of § 161. If in fig. 17 we now regard x and X as the ordinates to any point in the whole field of the figure, they will then be independent variables. It will now be better to name the vertical ordinate y, as X is throughout this book used to indicate a function dependent on x. The area (xy) will be a function of these two independent variables. Applying the law of § 161 to this function, we have

$$\left\{\frac{d(xy)}{dx}\right\} = \frac{\partial(xy)}{\partial x}\text{with } y \text{ constant} + \frac{\partial(xy)}{\partial y}\text{with } x \text{ constant} \times \frac{dy}{dx}$$
$$= y + x\frac{dy}{dx} = y + xy'$$

where the bracket { } indicates that the gradient is taken with concurrent change of x and y in the ratio given by y' on the right side. If the given y' be the x-gradient of the curve drawn in fig. 17, there is here reproduced the law of § 80, which is thus shown to be simply a particular case of a more general law, namely, that of § 161.

168. Definite Integral of Function of Independent Variables.—The equation of § 161 gives the increment of rise in level δh from any point of the surface to any other closely contiguous point. The integration of this increment of rise gives the total rise from one point to another point, near or distant, on the same surface. Taken between definite limits, this integral means the difference of level between two definite points on the surface. From any lower limit $n_1 w_1$ to any upper limit $n_2 w_2$ the definite integral is $(h_2 - h_1)$.

The **indefinite** integral is a general expression giving the height of the surface at **any** and **every** point measured from **any** convenient datum level.

169. Definite Integral of Function of Independent Variables.—In integrating from point 1 to point 2 (distant from each other), the integration may be followed out along a great variety of paths, the only condition a suitable path has to fulfil being that it must pass through both points 1 and 2. The path may be curved in any fashion, or be zigzagged in any regular or irregular manner. The integration along every such path will evidently give the same result. If in fig. 28 the points 1 and 2 be distant from each other, the integration might first follow the directly north path 1N, and then the directly west path N2. During the first part δw would be continuously zero, and the integration would extend from latitude n_1 to latitude n_2, keeping constantly to the longitude

w_1. During the second part, δn would be continuously zero. The same result is obtained by integrating first from 1 to W at constant latitude n_1, and then from W to 2 at constant longitude w_2.

170. Equation between Differences of Integrals.—Incidentally it may be noted that this gives, by converting the equation between the sums of these pairs of rises into an equation between the differences of the pairs of rises on opposite sides of the rectangle,

$$\left[\int_{n_1}^{n_2} \frac{\partial h}{\partial n} dn\right]_{w_1}^{w_2} = \left[\int_{w_1}^{w_2} \frac{\partial h}{\partial w} dw\right]_{n_1}^{n_2}$$

the left-hand expression meaning the difference between two integrations from latitude n_1 to latitude n_2, carried out along the meridians of longitude w_2 and w_1; while the right-hand similarly means the difference between two integrations each along a parallel of latitude and each between the same limits of longitude.

171. Indefinite Integral.—The **indefinite** integral h may be obtained by first integrating along **any** meridian up to an undefined point, and then from that same point along a parallel of latitude an indefinite distance; or the integration along the parallel of latitude may be effected first, to be followed by the meridional integration. In either case the second integration **must start from the same point as that at which the first finishes,** this point, however, being any whatever.

172. Independent Functional Integration Constants.—Although n and w may be varied quite independently, there is a relation between the law of the meridional section and that of the section of constant latitude which deserves notice. The equation of the meridional section, in which n is the variable, changes from section to section, *i.e.*, changes with the longitude. This equation, therefore, in general involves the longitude w. For any one such section the value of w entering into it remains constant. Thus the general expression for h may be taken as the sum of three terms, thus:—

$$h = \mathrm{N} + \mathrm{F}(n,w) + \mathrm{W}$$

where N is a function involving n but not w; W a function involving w but not n; and $\mathrm{F}(n,w)$ is the sum of such terms as involve both n and w. The partial gradient for any meridional section is

$$\frac{\partial h}{\partial n} = \mathrm{N}' + \mathrm{F}'_n(n,w)$$

W being a constant in this differentiation.

The partial gradient for any section of equal latitude is

$$\frac{\partial h}{\partial w} = W' + F'_w(n,w).$$

These two formulæ exhibit clearly the necessary relation between the two partial gradients. They differ, first, in N′ and W′, which are respectively functions of n alone and of w alone, and between which parts, therefore, there is complete independence; and, secondly, in $F'_n(nw)$ and $F'_w(nw)$, which are different but not independent, being necessarily related by the condition that they are the partial gradients of the same function involving both variables.

173. Independent Functional Integration Constants.—Written in terms of independent variables x, y, and χ the integral function of xy, these formulæ become

$$\chi = X + F(x,y) + Y$$

$$\frac{\partial \chi}{\partial x} = X' + F'_x(xy)$$

$$\frac{\partial \chi}{\partial y} = Y' + F'_y(x,y)$$

where X is a function of x only, and Y is a function of y only.

174. Complete Differentials.—In fig. 28 the slopes of the two lines 1W and N2 in the "North Elevation" give the westward gradient at the two latitudes n_1 and n_2. These lines are drawn parallel in fig. 28 because the points 1, 2 are close together, and for a **first degree of approximation** the difference of slopes through them may be neglected if the surface be continuous. If a second degree of approximation to accuracy be considered; that is, if we investigate "**second gradients,**" the difference between these two westward gradients must be taken into account. It is evidently

$$\frac{d}{dn}\left(\frac{\partial h}{\partial w}\right) . \delta n$$

and the difference between the rise from N to 2 and the rise from 1 to W is

$$\frac{d}{dn}\left(\frac{\partial h}{\partial w}\right) . \delta n . \delta w .$$

Similarly, the difference between the northward gradients W2 and 1N as seen in the "West Elevation" of the same figure is

$$\frac{d}{dw}\left(\frac{\partial h}{\partial n}\right) . \delta w$$

and the difference between the rise from W to 2 and the rise from 1 to N is

$$\frac{d}{dw}\left(\frac{\partial h}{\partial n}\right) . \delta w . \delta n .$$

But by § 170 these differences equal each other. Cancelling out the common product $\delta n . \delta w$, we have the equality

$$\frac{d}{dn}\left(\frac{\partial h}{\partial w}\right) = \frac{d}{dw}\left(\frac{\partial h}{\partial n}\right) .$$

Using the nomenclature of the end of § 172, since $\frac{dW'}{dn} = 0$, because W′ does not involve n, and similarly $\frac{dN'}{dw} = 0$, this equation becomes

$$\frac{d}{dn}\Big(F'_w(nw)\Big) = \text{say } F''_{nw}(nw) = \frac{d}{dw}\Big(F'_n(nw)\Big) = \text{say } F''_{wn}(nw) .$$

Thus it is indifferent whether the n or the w differentiation be taken first, and whether $F''_{nw}(nw)$ or $F''_{wn}(nw)$ be used as symbol.

Although these second-gradients, calculated in these two different ways, have the same value, they represent two perfectly distinct physical phenomena. The one is the *northward* rate of change of the *westward* gradient of h. The other is the *westward* rate of change of the *northward* gradient of h. That these are equal, whatever kinds of physical quantities be represented by h, n and w, is a proposition of mathematical physics that is most interesting and fertile in its various concrete applications.

175. Second x,y-Gradient.—When two functions of x and y fulfil the condition of being the partial x and y gradients of one and the same function, then the function formed by adding the products of these functions by δx and δy respectively, is said to be a **complete differential**. Thus if $\frac{\partial \chi}{\partial x}$ and $\frac{\partial \chi}{\partial y}$ be the functions, ascertained to be the partial x and y gradients of the same function χ, then

$$\frac{\partial \chi}{\partial x} . \delta x + \frac{\partial \chi}{\partial y} . \delta y$$

is a "**complete differential**," and this latter is said to be "**integrable.**" If this has been found, by accurate deduction from correct observation of physical fact, to be the increment of a real physical quantity, then it is certain that the function is theoretically integrable (although the integration may be impracticably difficult) and that its two parts will fulfil the condition of § 174. Of course, it is easy for the pure mathematician to invent functions of this sort that are not integrable, and incorrect physical observation or inaccurate deduction from physical investigation may lead to differentials that are not integrable; but such have no real physical meaning.

176. Double Integration by *dx* and *dy*.—Conversely, if any function of two independent variables, x, y, be twice integrated first by dx and then by dy, the result will be the same as if first integrated by dy and then by dx, being in either case the sum of a function dependent on both x and y and of two other functions depending separately, one of them on x alone and the other on y alone.

These two latter functions are introduced by the integrations in the same way as constants are introduced by integrations with respect to one variable; the one function being a constant with respect to one variable, and the other being a constant with respect to the other variable.

Thus, for example, if

$$\frac{d^2\chi}{dx\,dy} = a + bx + cy + e\,xy$$

then

$$\chi = \mathrm{X} + \mathrm{Y} + xy\left(a + \frac{b}{2}x + \frac{c}{2}y + \frac{e}{4}xy\right)$$

where the laws of the functions X and Y must be determined by "**limiting conditions.**"

The finding of χ from the given value of $\frac{d^2\chi}{dx\,dy}$ is called the double integration of this function, and is symbolised by

$$\iint \frac{d^2\chi}{dx\,dy}\,dx\,dy$$

or $\iint \phi(xy)dx\,dy$ if $\phi(xy)$ be the given functional form of $\frac{d^2\chi}{dx\,dy}$.

177. Graphic representation of Double Integration by *dx* and *dy*.—The meaning of the double integration of $\phi(xy)$ may be represented graphically in the following different manner.

Let $\phi(xy)$ be represented by the height of a surface from a datum plane, the co-ordinates parallel to this plane being x and y.

Then the first integration $\int \phi(xy)dx$ may be considered as extending along a section perpendicular to the datum plane and parallel to the x-axis, in this integration y being a constant. The result of this integration is a general formula for the area of any such section. Two such sections at the very small distance δy apart will inclose between them, under the surface and above the datum plane, a volume equal to δy multiplied by the area of the y-constant section at the middle of δy. This volume is, therefore, $\left\{ \int \phi(xy)dx \right\} . \delta y$, and the whole volume under the surface and above the datum plane therefore properly represents $\iint \phi(xy)dxdy$. This geometric conception is more easily grasped if the integration be taken between limits.

178. Connection between Problems concerning One Independent Variable and those concerning Two Independent Variables.—In an investigation concerning two mutually dependent variables, such as those in Chapters I. to VII., the two variables may always be represented by the co-ordinates to a plane curve. This curve may be looked on as a plane section of a surface, the three co-ordinates to the points upon which are related to each other by the more general kind of law dealt with in this chapter. Thus the former problems may always be conceived of as the **partial** solutions of more general laws connecting three variables with only one specific relation between them. The problems of Chapters I. to VII. may thus be considered **special cases** of more general problems of the kind now dealt with, and each of them might be deduced by specialising from a more general theorem.

CHAPTER IX.

MAXIMA AND MINIMA.

179. General Criterions.—In fig. 1, at the parts C, E, H, K, R S, U, the l-gradient of h is zero. The points C and K are places where h rises to a maximum, the maximum K being greater than the maximum C, but the phrase "**maximum**" being understood to mean a value greater than any neighbouring value on either side. E is a place where h falls to a minimum.

Thus the gradient falls to zero wherever there is either a maximum or a minimum value.

At the maxima points, C and K, the forward gradient passes through zero by changing from positive to negative, that is, the increase of the gradient is negative at these places.

At the minimum point E, the forward gradient changes from negative to positive, so that its increase is positive.

Thus the criterion for distinguishing between a maximum and a minimum is, that at the former the **second gradient** or second differential coefficient is negative, while at a minimum point it is positive.

It is not always, however, necessary to find the sign of the second gradient in order to make sure whether the point is a maximum or a minimum. For instance, if it be known that at the place where the first gradient is zero, the value of h is positive, and if it be also known that at two points near and on either side of this place the value of h becomes zero, or of any positively less value than at this place of zero gradient, then evidently this place gives a maximum.

At the place H, fig. 1, the second gradient is zero, because to the left of H it is negative, while to the right of H it is positive. This case of zero second-gradient occurring along with zero first-gradient is the limiting case coming in between the two previous ones, giving respectively maxima and minima; and it gives neither a maximum nor a minimum. This includes the case of the dead level RS, where also both first and second gradients are zero.

Usually one's general knowledge of the physical phenomenon being investigated is sufficient, without need of evaluating the second gradient, to indicate whether or not there is any such point as H. That is, the practical man who thinks of what he is working at, and does not follow blindly mere mathematical formulas, runs substantially no risk of mistaking such a point as H for either a maximum or a minimum point.

180. Symmetry.—In very many practical problems conditions of symmetry show clearly where a maximum or minimum occurs without the need of investigating either first or second gradient. Thus, if a beam be symmetrically supported, symmetrically loaded, and have a symmetrical variation of section on either side of a certain point of its length, which point is then properly called its centre, then the bending moment and the deflection each reach a maximum at this centre. Such considerations are to be utilised wherever possible, and their use is sometimes more profitable in practical result than the more strictly mathematical process.

181. Importance of Maxima in Practical work.—As examples of the utility of these theorems may be cited the finding of the positions and magnitudes of maximum bending moments, of maximum stresses, of maximum deflections, of maximum velocities, of maximum accelerations of momentum, of the positions of rolling load on bridges to give maximum stress in any given member of the bridge, etc., etc. All these things are of special importance in the practical theory of engineering. In the jointing of pieces together in machines and static structures, it is never possible to obtain uniform stress over the various important sections of the joint. It is of the greatest importance to find the maximum intensities of stress on such sections, because the safety of the construction depends on the maximum, hardly ever upon the average, stress. The average stress on the section is found by dividing the whole load on the section by the whole area of the section. Such average stresses are often very different from the maximum stress, and no reliance ought to be placed upon them as measures of strength and safety.

Another class of technical problems in which maxima points are of paramount importance is that in which two or more sets of variable driving efforts, or of variable resistances, are superimposed in a machine. Thus a first approximation to the turning moment on the crank shaft of a steam engine of one cylinder, makes this moment vary as $\sin a$, where a is the angle at which the crank stands from the dead point. If there be two cylinders in which the total steam pressures are P_1 and P_2, constant throughout the stroke, and the two cranks, keyed on the same shaft, stand apart by an angle A; then a being the angle from dead point of one crank, $(a+A)$ is that of the other. A remains constant while a varies. If S_1 and S_2 be the two strokes, the total turning moment on the shaft is

$$\tfrac{1}{2}\{S_1P_1 \sin a + S_2P_2 \sin (a+A)\}$$

which reaches a maximum when its a-gradient is zero; that is, when

$$\frac{\cos a}{\cos(a+A)} = -\frac{S_2P_2}{S_1P_1}.$$

This ratio is *minus* unity when $S_2P_2 = S_1P_1$; and if, further, $A = 90^\circ$, then $a = 45^\circ$ at the maximum.

182. Connecting Rod Bending Moments.—The connecting rod of an engine is at each instant bent by transverse accelerations of momentum, which, taken per inch length, would increase uniformly from zero at the crosshead to a certain amount at the crank end if the section of the rod were uniform. The actual bending moments on the rod follow nearly the law due to this distribution of load, because the excess of weight in each head is approximately centred at the point of support at either end and, therefore, does not affect the bending moments.

If L be the whole length; l the length to any section from the crosshead; w the transverse load per inch at the crank end: then $\frac{wl}{L}$ is the load per inch at l. On the section at l, therefore, the bending moment is $\left\{ \frac{wL}{6} . l - \frac{wl^2}{2L} . \frac{l}{3} \right\}$.

The first l-gradient of this is zero at the point of maximum moment; that is, this point has a distance l given by

$$\frac{wL}{6} - \frac{wl^2}{2L} = 0$$

or

$$l = \frac{L}{\sqrt{3}} = \cdot 5773L.$$

Inserting this value of l in the general value of the moment, we find as the maximum moment

$$\left\{ \frac{wL}{6} . \frac{L}{\sqrt{3}} - \frac{wL^3}{L6 \times 3\sqrt{3}} \right\} = \cdot 06415wL^2$$

which may be compared with $\cdot 0625wL^2$, which is the central moment in the case of the same total load, $\frac{1}{2}wL$ being uniformly distributed along the whole span. It is $2\frac{1}{2}\%$ greater than this latter, and its position is $7\frac{1}{2}\%$ of the span away from the centre.

183. Position of Supports giving Minimum Value to the Maximum Bending Moment on a Beam.—The following illustrates how maxima of arithmetic, as distinguished from algebraic, quantities may sometimes be found without use of a differential coefficient. If a beam, freely supported, overhang its supports equally at the two ends, and be uniformly loaded; then certain positions for the supports will make the maximum moment less than for any other positions of these supports.

Let L and w per inch be the total length and the load, and l the span between the supports. The bending moment on the section over each support is $w . \frac{L - l}{2} . \frac{L - l}{4} = \frac{w}{8}(L - l)^2$. The central moment, taking it of opposite sign, is

$$\frac{wL}{2} . \frac{l}{2} - \frac{wL}{2} . \frac{L}{4} = \frac{wL}{4}\left(l - \frac{L}{2}\right).$$

If this latter be negative, *i.e.*, if $l<\frac{L}{2}$, these two moments will be of the same physical sign; that is, the beam will be bent convex on its upper surface throughout its whole length. If $l>\frac{L}{2}$, a certain central length will be concave on the upper surface, and inside and outside this length the moments will be of opposite sign. As l is made larger, the magnitude of the central moment becomes always larger and that of the moment at the supports always smaller. Therefore, neither has any algebraic maximum. But when they are arithmetically equal, their common arithmetic magnitude is then less than the magnitude of the greater of the two for any other span. So that, irrespective of sign, the minimum of the arithmetic magnitudes of the three maximum moments is reached when

$$\frac{w}{8}\left(L-l\right)^2=\frac{wL}{4}\left(l-\frac{L}{2}\right)$$

or

$$l=\cdot 5858L \text{ and } \frac{L-l}{2}=\cdot 2071L.$$

Inserting this value of l in either formula for the moment we find

Central moment = moment over each support = $\cdot 02144wL^2$

which is only 17% of the central moment on the same beam with same load when supported at the two ends. This fact may be regarded as the basis of the great economy of the modern "**cantilever**" style of bridge building.

184. Position of Rolling Load for Maximum Moment and for Maximum Shear.—The next example shows how reasoning about increments, instead of differential coefficients, may be used to find maximum values.

The bending moment produced by a load on any section of a girder, supported freely at its ends, is of the same sign wherever the load be placed within the span. Therefore the moment on each and every section produced by a uniform rolling load reaches a maximum when the load covers the whole span.

The right-handed integral shear stress on each section equals the supporting force at the left-hand support, *minus* the load applied between this support and the section. Therefore, any load applied right of the section increases this shear stress, because it increases the left supporting force and leaves unaltered the load between it and the section. But a load applied left of the section decreases the same stress, because it increases the left supporting force less than it increases the load between it and the section. Therefore the right-handed shear stress on any section due to a uniform rolling load reaches a maximum when the load covers the whole of that part of the span to the right of the section, but covers none to the left of it. The left-handed shear stress reaches its maximum when the part left of the section is covered. The arithmetic maximum of the stress is reached when the larger of the two segments into which the section divides the span is covered while the shorter is empty.

185. Most Economical Shape for I Girder Section. — The economic proportioning of sections is illustrated by the following.

Let M be the bending moment strength of an I girder, whose depth is H outside the flanges and h inside them, and whose flange breadth is B and web thickness wB.

Let the area of its cross-section be called S.

Then the moment strength per square inch of section may easily be shown to be

$$\frac{M}{S}=\frac{k}{6H}\cdot\frac{H^3-(1-w)h^3}{H-(1-w)h}.$$

If H and h be increased in the same proportion, this $\frac{M}{S}$ will increase in proportion to the first power of either of them, large sections being always stronger and stiffer per square inch than small ones. It also increases if H is increased without alteration of h. $\frac{M}{S}$ also increases as w is decreased towards zero, the web section contributing to the moment strength less than the flange section and, therefore, less than the average for the whole section. If, however, the web thickness be supposed fixed in accordance with the requirements of shear strength, and if h be diminished while H is unaltered, thereby thickening the flanges *internally*, this flange thickening will, up to a certain limit, increase the economy of the section, beyond which a further thickening will decrease it again. The h-gradient of $\frac{M}{S}$ is

$$\frac{k}{6H}\cdot\frac{-3(1-w)h^2\{H-(1-w)h\}+(1-w)\{H^3-(1-w)h^3\}}{\{H-(1-w)h\}^2}.$$

If this be equated to zero, there results

$$2(1-w)\left(\frac{h}{H}\right)^3-3\left(\frac{h}{H}\right)^2+1=0.$$

This is a cubic equation giving the most economical depth inside the flanges when that outside the flanges, as also the ratio w of web thickness to flange width, are fixed by other considerations. This ratio between h and H essentially depends on w; if $w=0$, giving zero thickness to the web, the above equation gives $h=\text{H}$, *i.e.*, gives zero thickness to the flange also, or the whole section shrinks to zero area. When $w=\cdot5$, it gives $h/\text{H}=\cdot6527$. A useful exercise for the engineering student is to solve this equation for values of w ranging up to ·5. The solution can be very easily effected by the method of solving for w taking a series of values of h/H ranging from 1 down to ·6; tabulating these graphically as a curve; and then reading from the curve the h/H for any desired values of w.*

186. Most Economical Proportions for a Warren Girder.— The economic proportioning of general dimensions is the subject of the next example.

If a Warren girder of height H, and length of bay B, have the bay width B made up of b the horizontal projection of a tie-brace and $(\text{B}-b)$ the horizontal projection of a strut brace; then the weight of material G required to give the structure strength to carry the desired load, exclusive of that spent in jointing the various members together, may be expressed by a formula of six terms involving, besides H, B and b, also the span, the load, the stresses allowed on the sections, and four numerical coefficients which do not vary with the H nor with the span or load nor with the ratio $\frac{b}{\text{B}}$ and vary very little with the number of bays. The same formula may be applied to any pattern of lattice girder by suitably adjusting the numerical coefficients.

Four terms of this weight decrease as H increases, while two increase. A certain girder depth H will, therefore, be most economical in expenditure of material. Assuming everything but H to remain constant, and equating the H-gradient of the above to zero, there is obtained the best girder depth.

Again, the girder weight contains two terms increasing with B and two others decreasing with B. Assuming the ratio $\frac{b}{\text{B}}$ and all other quantities except B to be kept unaltered and equating the B-gradient of G to zero, we obtain a formula for the most economical bay width for given span and height, which gives also indirectly the best number of bays to insert in the given span. This formula cannot, however, be precisely followed, because the number of bays must necessarily be a whole number while the equation gives in general a fractional number.

The girder weight also varies with b in two terms, one of which

* See Appendix H.

increases while the other decreases with b. Considering everything but b as fixed, and putting the b-gradient equal to zero, a rule is found for proportioning the length of the ties to that of the struts.

These results are not formulated here because to guard against their incorrect application requires rather more explanation of special bridge-building detail than is suitable to this treatise.

187. Minimum Sum of Annual Charge on Prime Cost and of Working Cost.—Very many technical problems are, or ought to be, solved by reducing to a minimum the sum of two main costs: first, the initial cost of construction and other necessary preliminary expenses; second, the cost of working, maintenance, and repair. These can only be added when reduced to terms rationally comparable, and this is usually done by reducing both to an **annual** cost or charge. Interest on all initial expenses, including prime cost of actual construction, is to be added to an annual charge to provide for a sinking fund to reproduce the capital after a period within which it is estimated that the plant will become useless from being worn out, or having become obsolete —which annual charge is often referred to as "**depreciation**"— and this forms the first part of the whole cost. The second part consists of wages, materials used up in working, power for driving, etc. If the initial expenditure be skilfully and wisely spent, its increase nearly always, within limits, decreases the working expenses. It follows that in most if not all cases a certain initial expenditure is that that will make the total annual cost a minimum. Thus the adoption of a larger ratio of expansion in a steam engine will, within certain limits, diminish the consumption of water and of coal required to produce any required horse-power; but it will necessitate a larger and more expensive engine for this same horse-power, which will be, moreover, more costly to keep in good working order; and this is the real consideration which ought to determine the commercially most economic cut-off in steam engines. Lord Kelvin's calculation of the best cross-sectional area of electrical leads is another example of this kind of problem.

188. Most Economical Size for Water Pipes.—The following is a similar example directed to the calculation of the most economic diameter of water pipes, first published by the author in 1888.

If a given weight or volume of water is to be delivered per hour at a certain station at a certain pressure, this means the same thing as delivering so much water horse-power at this station. Let this horse-power be called H, and the pressure demanded at the point of delivery p; let L be the distance from the pumping or gravity-power station, and d be the internal diameter of the pipe. Then the loss of power in transmission, through friction and viscosity (exclusive of loss at bends and valves), can be shown to be nearly $a\frac{LH^3}{p^3d^5}$, where a is a numerical coefficient dependent on the smoothness of the inside surface of the pipe and on the shape of cross section. If q be the cost per hour of generating 1 horse-power, and if the delivery be continued for T hours per year; then the cost of this waste horse-power per year is

$$qTa \cdot \frac{LH^3}{p^3d^5}.$$

The prime cost of pipes and pipe-laying (including trenching) may be taken as the sum of two terms, the first proportional to the length L, and independent of the size of pipe; the second proportional to the quantity of metal in the pipe. The thickness of pipe requires to be designed according to the formula $\left(A + \frac{pd}{B}\right)$ where A and B are constants. The part of the initial cost which varies with the diameter will, therefore, give an annual cost in interest and depreciation of

$$rLd\left(A + \frac{pd}{B}\right)$$

where r is a factor dependent on (1) the price of iron; (2) the nature of the ground to be trenched; and (3) the prevailing rate of interest on money.

That part of the total annual cost which varies with the size of the pipe is, therefore,

$$qTa\frac{LH^3}{p^3d^5} + rLd\left(A + \frac{pd}{B}\right).$$

Equate the d-gradient of this to zero; the result is

$$\left(A + 2\frac{pd}{B}\right)d^6 = 5 \cdot \frac{qa}{r} \cdot T \cdot \left(\frac{H}{p}\right)^3$$

an equation for the determination of d, which, although of the 7th degree, is very easily solved with the help of Barlow's Tables of powers. Other things being equal, it makes the diameter vary according to a power of $\frac{H}{p}$ lying between $\frac{1}{2}$ and $\frac{3}{7}$.*

* After demonstrating this law, the author incidentally discovered that

189. Maximum Economy Problems in Electric Transmission of Energy.—Calculations of maxima enter into many important problems in electric engineering. Let an electric current generator exert an E.M.F. equal to E, and that part of the electric resistance of the circuit under the control of the supply company be R, while the rest of the circuit has a resistance r. Then the current is $\frac{E}{R+r}$ if there be no counter E.M.F. The electric work per second in the external resistance r is $\frac{E^2 r}{(R+r)^2}$. If it be desired to make this as great as possible by adjusting r without alteration in either E or R, we find the r-gradient of this external work to be zero when $r=R$, and the external work is then $\frac{E^2}{4r}$, the whole work being double this.

If there be an external counter E.M.F. equal to e, the current is $\frac{(E-e)}{R+r}$ and the work done on the counter E.M.F. is $\frac{(E-e)e}{R+r}$. This work continuously decreases as either r or R is increased without change in E or e. If E, R, and r be fixed while e is adjustable, the e-gradient of this work is $\frac{E-2e}{R+r}$, which becomes zero when $e=\frac{1}{2}E$, giving the maximum work that can be done on e under these conditions equal to $\frac{E^2}{4(R+r)}$ and the efficiency $\frac{e}{E}$ equal to $\frac{1}{2}$. This gives maximum motor work under the prescribed conditions, *not maximum efficiency.*

In the early days of attempted transmission of power by electric current, it was believed by many that this showed that no higher efficiency than $\frac{1}{2}$ could be reached in such transmission. This misapprehension was acted upon practically by electricians of good reputation, and retarded progress in this branch of engineering.

The efficiency of the transmission is $\frac{e}{E}$, and this can be made nearly unity by making e nearly equal to E. By so doing, the excess of driving over driven E.M.F., namely $(E-e)$, is diminished, and the current and horse-power delivered are diminished unless the decrease so effected be neutralised by increase in e, and, therefore, also in E, or by decrease in $R+r$. With small resistances, how-

an old established firm of water engineers had empirically framed a rule which was nearly identical with the above.

ever, combined with high voltages, large horse-powers can be transmitted with high electric efficiency. The necessary voltages and resistances required to deliver any required horse-power with any given efficiency are easy to calculate, but the calculations do not illustrate the subject of this chapter.

A large number of electric transmission calculations of values giving maximum economy, etc., under various conditions, may be found in a series of articles by the author in *Industries*, 1889.*

190. Maxima of Function of Two Independent Variables.— If a function of two independent variables be represented by the height of a surface with the two variables as horizontal co-ordinates, then any plane vertical section will give a curve which will rise to maximum height or fall to minimum height where the partial first gradient is zero, and the partial second gradient is not zero. If two such sections cross each other at a surface point where the partial first gradients are zero in both sections, then the surface is level throughout a small extent all round the intersection point. There are six cases to be distinguished.

(1) The partial second gradient is positive in both sections; then the point is at the bottom of a hollow in the surface, all neighbouring points being higher; so that the value of the function represented by the height is here a minimum.

(2) The partial second gradients in the two sections are both negative; then the point is at the top of a convexity or globular part of the surface, all neighbouring points being lower; so that the value of the height function is here a maximum.

(3) The one partial second gradient is positive, while the other is negative. Here the surface is anticlinal, or saddle-shaped; it is hollow in one direction, and round in the other. This gives neither minimum nor maximum value to the height function.

(4) The one partial second gradient is positive, while the other is zero, passing from positive to negative. Here the part of the surface is the junction between a hollow portion lying on one side of a vertical plane and an anticlinal portion lying on the other side of the same; and the height function is again at neither maximum nor minimum value.

(5) The one may be negative, while, as in (4), the other is zero.

* Also many other problems dealing with modern conditions are solved in Chap. III., on "Economic Deductions from Statistics and Technical Conditions," in the author's *Electric Traction*, published 1905.

There is here indicated the junction between a round portion and an anticlinal portion; and again neither maximum nor minimum occurs here.

(6) Again both may be zero. Here two semi-anticlinal surfaces join together, and the point gives neither maximum nor minimum.

The first two cases alone are important as regards finding maximum and minimum values. In these cases both first gradients are zero, and the second gradients are of the same sign. To find the maximum or minimum values of a function of two independent variables, the process is to equate the two partial first gradients to zero, and combine these two equations as simultaneous ones. Afterwards the second gradients should be examined as to sign if the physical character of the problem is not so plain as to make this unnecessary.

191. Most Economic Location of Junction of Three Branch Railways.—As an illustration of this process, take an elementary problem in the theory of railway location. Three centres of traffic are supposed situated in a plain across which the construction of the railway is equally easy in all directions. The three centres are to be joined by three branch lines radiating from a junction, the finding of the best position for which junction is the problem proposed. The traffic issuing from and entering each centre is the sum of the two traffics between it and the other two (inclusive possibly of traffic going *through* these to more distant points). The importance of the three traffics to and from the three centres being properly measured, and being here symbolised by A, B, and C; and the distances of the three centres from the junction being called a, b, c; the correct solution of this problem means the location of the junction so as to give a minimum value to

$$Aa + Bb + Cc$$

A, B, C being given, while a, b, c are to be found.

Referring to the notation of fig. 29, in which the distance between the points of traffic A and C is called L, and the projection on L of the distance between the points of traffic A and B is called λ, while the projection of the same perpendicular to L is called H; the problem may be conveniently stated to be to find the co-ordinates l and h of the best junction.

We have with this notation

$$a = \{l^2 + h^2\}^{\frac{1}{2}} \qquad b = \{(\lambda - l)^2 + (H - h)^2\}^{\frac{1}{2}} \qquad c = \{(L - l)^2 + h^2\}^{\frac{1}{2}}.$$

The two partial l and h gradients of $(Aa + Bb + Cc)$ are taken; thus

$$\frac{\partial a}{\partial l} = \frac{l}{(l^2+h^2)^{\frac{1}{2}}} = \frac{l}{a} = \cos\alpha$$

$$\frac{\partial b}{\partial l} = -\frac{\lambda - l}{b} = -\cos\beta$$

$$\frac{\partial c}{\partial l} = -\frac{\mathrm{L} - l}{c} = \cos\gamma$$

$$\frac{\partial a}{\partial h} = \frac{h}{(l^2+h^2)^{\frac{1}{2}}} = \frac{h}{a} = \sin\alpha$$

$$\frac{\partial b}{\partial h} = -\frac{\mathrm{H} - h}{b} = -\sin\beta$$

$$\frac{\partial c}{\partial h} = \frac{h}{c} = \sin\gamma$$

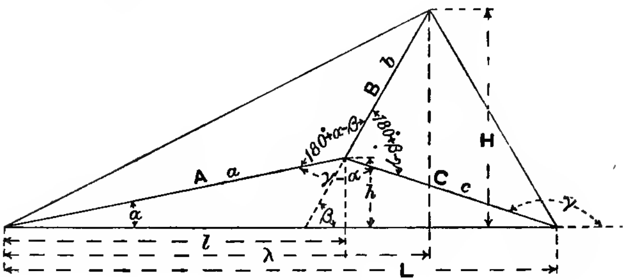

Fig. 29.

Therefore equating both partial gradients to zero, we obtain the simultaneous equations

$$\left.\begin{array}{l}\mathrm{A}\cos\alpha - \mathrm{B}\cos\beta + \mathrm{C}\cos\gamma = 0\\ \mathrm{A}\sin\alpha - \mathrm{B}\sin\beta + \mathrm{C}\sin\gamma = 0\end{array}\right\}.$$

Eliminating the terms in B by cross-multiplication, and again similarly eliminating the terms in C, there are obtained the two ratios

$$\frac{\mathrm{A}}{\mathrm{C}} = \frac{\sin(\gamma-\beta)}{\sin(\beta-\alpha)} = \frac{\sin(180^\circ+\beta-\gamma)}{\sin(180^\circ+\alpha-\beta)}$$

and

$$\frac{\mathrm{A}}{\mathrm{B}} = \frac{\sin(\gamma-\beta)}{\sin(\gamma-\alpha)} = \frac{\sin(180^\circ+\beta-\gamma)}{\sin(\gamma-\alpha)}.$$

By reference to the figure it will be seen that the three angles entering into these ratios, namely, $(180^\circ+\beta-\gamma)$, $(180^\circ+\alpha-\beta)$, and $(\gamma - \alpha)$, are the three angles between b and c, a and b, c and a. Since the sides of a triangle are proportional to the sines of the opposite angles, it follows immediately that these three angles $\widehat{bc}$, $\widehat{ab}$, and $\widehat{ca}$, to which the three branches are to be adjusted, are the exterior angles of a triangle whose sides are made equal (to any convenient scale) to A, C, and B. By constructing this triangle these angles can be found, and by drawing upon two of the lines joining two pairs of centres of traffic two arcs of circles containing these angles, the proper junction is located as the intersection of these arcs.

This result may be perceived more directly by noticing that the two partial gradients to be equated to zero are, one the sum of the projections on L, and the other the sum of the projections on H, of A, B, C, measured along the lines a, b, c, outwards from the junction. Other maximum problems of technical interest are solved in Appendices I., K., L.

CHAPTER X.

INTEGRATION OF DIFFERENTIAL EQUATIONS.

192. Explicit and Implicit Relations between Gradients and Variables.—The utility of the art of integration arises from the fact that in the investigation of phenomena it often happens that the discovery of the ratio between simultaneous increments (or gradient) of mutually related quantities is effected more easily by direct observation than is the discovery of the main complete relation between these same quantities. This complete relation is then logically deduced by the help of integration, combined with the further observation of special or "**limiting**" values. The complete relation being less general than the differential relation, there appears in it an "**arbitrary constant**" whose value is not given by the differential relation, and which value must be discovered by examination of the "**limiting conditions.**" The differential relation applies equally well to a whole "**family**" of integral relations which differ among themselves in respect of these limiting conditions.

If, when the differential relation is expressed as an equation, the gradient can be placed alone on one side of the equation, while on the other appears a function of **one only** of the mutually dependent variables; then, in order to establish the integral relation, nothing more has to be done than to integrate this latter function **directly** according to one or other of the methods explained in previous chapters, or given in the appended *Classified Reference List of Integrals*. This is the case of the gradient being expressed as an **explicit function of one of the variables.**

If, however, the differential equation involve the gradient and both variables in such a way that the above simple separation does not appear; that is, if the gradient appear as an **implicit** function of the variables; then either the implicitness of the relation must be got rid of by some algebraic process, or else a special method of integration must be employed. This process is called the "**solution**" **of the differential** equation.

193. Degree and Order of an Equation. Nomenclature.—An ordinary algebraic equation is said to be of the n^{th} **power** or **degree** when it involves the n^{th} power of the "**unknown**" quantity or of either of the "**variables.**" In a differential equation, the gradients being the quantities to be got rid of by integration, the equation is said to be of the n^{th} **degree** when the n^{th} power of the highest occurring gradient is involved in it.

If it involve a **second** gradient $\left(\frac{d^2X}{dx^2}\right)$, it is said to be of the **second order.** If it involve an n^{th} gradient, it is said to be of the n^{th} **order.**

The integral relation deduced from the differential equation may be called the **integral equation,** but is also called the **primitive equation.**

In what follows X′ will mean $\frac{dX}{dx}$;

$$\text{and } f'(x) \equiv \frac{df(x)}{dx};\ f'(X) \equiv \frac{df(X)}{dX} - \textbf{not } \frac{df(X)}{dx};\ f'(X') \equiv \frac{df(X')}{dX'}.$$

194. $X' = f(x)$.

The form of differential equation simplest to integrate is the **explicit** relation,

$$X' = f(x)$$

where $f(x)$ is a function involving the variable x alone, and not X. This is equivalent to

$$dX = f(x)dx$$

and if $f(\ )$ is integrable by help of any of the formulas in the Reference List, the integration can be effected at once; thus:—

$$X = \int f(x)dx + C$$

where C is the constant of integration.

195. $X' = f(X)$.

Again, if $$X' = f(X)$$

where $f(X)$ involves X only and not x, then

$$\frac{dX}{f(X)} = dx$$

and direct integration of $\frac{1}{f(\,)}$, if it be possible by one of the formulas in the Reference List, gives the integral relation between X and x

$$\int \frac{dX}{f(X)} = x + C.$$

As an example, if $f(X) = X$; then $\log_e X = x + C$.

As another example, if $f(X) = \sin X$; then $\log_e \tan \frac{X}{2} = x + C$.

196. $X' = f(x)F(X)$.

Again, if the differential relation be found in the form

$$X' = f(x)F(X)$$

where $f(x)$ is a function involving x and **not** X,
while F(X) ,, ,, X ,, x;
then

$$\frac{dX}{F(X)} = f(x)dx$$

and

$$\int \frac{dX}{F(X)} = \int f(x)dx + C$$

which gives the integral solution if both these integrals can be found directly.

As an example, take

$$X' = \cos x \sin X;$$

then

$$\log_e \tan \frac{X}{2} = \sin x + C.$$

The case of § 195 is only a special case of the more general method of the present article, namely, the case in which $f(x) = 1$; while that of § 194 is the other special case in which $F(X) = 1$.

In all these cases X′ has been found as a function of x and X in a form in which X′, x, and X can be completely separated as distinct terms or factors in the equation.

197. $x=f(X')$.

Suppose now that a certain function of the gradient X′ is found to equal x; thus,

$$x=f(X').$$

This equation may be capable of easy algebraic solution so as to give X′ as an explicit function of x; thus,

$$X'=f^{-1}(x)$$

where $f^{-1}(\,)$ means the function that is the inverse of $f(\,)$. It must be noted that such a solution has in general more than one root. Thus, if $f(X')$ be a quadratic rational function of X′, there are two roots. This solution can then be dealt with by § 194; thus,

$$X=\int f^{-1}(x)dx+C$$

provided the function $f^{-1}(\,)$ turns out to be directly integrable.

This method, however, may be impracticable, or else may involve more labour than the following.

Take the X-gradient of

$$x=f(X')$$

$$\frac{dx}{dX}=\frac{1}{X'}=f'(X')\frac{dX'}{dX}$$

from which we deduce by transposition

$$dX=X'f'(X')dX'$$

and

$$X=\int X'f'(X')dX'+C.$$

Here $\int X'f'(X')dX'$ is the same function of X′ as $\int xf'(x)dx$ would be of x; and if $xf'(x)$ is directly integrable by dx, the above can be directly found as a function of X′. Let this function be called $\phi(X')$, and suppose it expressed in terms of X′. Then from the two simultaneous equations, of which the first is the original differential equation,

$$\left.\begin{aligned} x&=f(X') \\ \text{and}\quad X&=\phi(X')+C \end{aligned}\right\}$$

X′ can be eliminated so as to leave an equation involving only x and X. This is the integral solution of the given differential equation.

As an example let

$$x = \sin X'.$$

Then

$$f'(X') = \cos X'$$

and

$$\int X' \cos X' dX' = X' \sin X' + \cos X'$$
$$= xX' + (1 - x^2)^{\frac{1}{2}} = x \sin^{-1} x + (1 - x^2)^{\frac{1}{2}}.$$

Therefore

$$X = x \sin^{-1} x + (1 - x^2)^{\frac{1}{2}} + C.$$

The same result is obtained by solving the given differential equation for X', viz., thus $X' = \sin^{-1} x$, and integrating directly from this.

198. $X = f(X')$.

If the implicit relation is found in the form

$$X = f(X')$$

either the algebraic solution of this for X' may be obtained, whence the integration

$$\int \frac{dX}{f^{-1}(X)} = x + C;$$

or else a method similar to that of last paragraph may be followed. Thus, taking the x-gradient,

$$X' = f'(X') \frac{dX'}{dx};$$

whence

$$\int \frac{f'(X')}{X'} dX' = \text{say } \phi(X') = x + C$$

this integration by dX' involving X' only, and giving some function of X', which is here symbolised by $\phi(X')$.

From the two simultaneous equations

$$\left.\begin{aligned} X &= f(X') \\ \text{and} \quad x + C &= \phi(X') \end{aligned}\right\}$$

X' is to be eliminated by ordinary algebraic means, leaving the integral equation involving only x and X.

199. $mX = X'$.

A particular case of the last is that of $f(X') = \dfrac{1}{m} X'$. Here

$$X = \frac{1}{m} X' = \frac{1}{m} \frac{dX}{dx}$$

or

$$dx = \frac{1}{m} \frac{dX}{X}$$

the integration of which gives directly

$$x = \frac{1}{m} \log_e (CX) .$$

In fact, in this case the differential formula of § 198 reduces to that of § 195, and is the first of the two examples of the result of § 195 given in that paragraph.

200. $X = xf(X')$.

A differential formula only slightly different from that of § 198, and to be dealt with in the same general manner, is

$$X = xf(X') .$$

Taking the x-gradients of both sides,

$$X' = f(X') + xf'(X') \frac{dX'}{dx}$$

from which

$$\frac{f'(X')}{X' - f(X')} dX' = \frac{dx}{x}$$

and therefore

$$\int \frac{f'(X')}{X' - f(X')} dX' = \text{say } \phi(X') = \log (Cx) .$$

From the two simultaneous equations

$$\left. \begin{aligned} X &= xf(X') \\ \text{and} \quad \log (Cx) &= \phi(X') \end{aligned} \right\}$$

X' is to be eliminated algebraically so as to leave the integral equation between x and X.

201. $X = nxX'$.

A particular case of the last formula, which is also at the same time a particular case of § 196, is

$$f(X') = nX'$$

or

$$X = nxX'$$

or

$$\frac{dX}{X} = \frac{1}{n} \frac{dx}{x}$$

whence by direct integration of each side

$$X = Cx^{\frac{1}{n}}.$$

202. $X = \pm xX' + f(X')$.

If the differential equation be of the more involved form

$$X = xX' + f(X'),$$

by taking the x-gradients on both sides, there is obtained, X' cancelling out from the two sides,

$$0 = \{x + f'(X')\}X''.$$

This equation has two solutions. The first is

$$X'' = 0$$

whence

$$X' = C_1 \text{ and } X = C_1 x + C_2$$
$$= C_1 x + f(C_1)$$

since $X = xX' + f(X')$ and $X' = C_1$. This is a **partial** integration of the given differential equation.

The other solution is

From this and
$$\left.\begin{aligned} x + f'(X') &= 0 \\ X = xX' + f(X') \end{aligned}\right\}$$

treated as simultaneous equations, X' may be algebraically eliminated, leaving an equation giving X in terms of x either explicitly or implicitly. Let this equation be symbolised by $\phi(x,X) = 0$. This $\phi(x,X) = 0$ is a second **partial** solution of the given differential equation.

The combination of these two partial solutions gives the complete solution in its most general form, which, in application to whatever physical problem may be in hand, must be particularised by the insertion of the "**limiting conditions.**" These limiting conditions sometimes exclude one of the "**partial**" solutions as impossible, leaving the other partial solution as the full true solution of the particular physical problem in hand.

The solution of a more generalised form of this differential equation is given in § 210, the method of solution depending on that of § 208.

A form differing from the last only in the sign of xX' is

$$X + xX' = f(X').$$

Here $X + xX'$ is the x-gradient of xX.

Therefore, the integration gives

$$x\mathrm{X} = \int f(\mathrm{X}').dx = \int \frac{f(\mathrm{X}')}{\mathrm{X}''}.d\mathrm{X}';$$

and, if X'' be expressible in terms of X' alone and the function $\frac{f(\mathrm{X}')}{\mathrm{X}''}$ be integrable by $d\mathrm{X}'$, this integration will give an equation between x, X, and X', between which and the original equation, X' may be algebraically eliminated, leaving one involving only x and X.

203. Homogeneous Rational Functions. — If the relation between x, X, and X' be found in the form

$$(ax^m + bx^{m-1}\mathrm{X} + cx^{m-2}\mathrm{X}^2 + \text{----})\mathrm{X}' = \mathrm{A}x^m + \mathrm{B}x^{m-1}\mathrm{X} + \mathrm{C}x^{m-2}\mathrm{X}^2 + \text{----}$$

where the (x,X) functions on the two sides are both "**homogeneous**" of the m^{th} degree, that is, where each consists of a series of products of powers of x and X, the sum of the two indices in each term being m; then by dividing each side by x^m, this may be converted into an equation in $\frac{\mathrm{X}}{x}$.

$$\text{Call } \frac{\mathrm{X}}{x} \equiv \mathfrak{X}, \text{ or } \mathrm{X} \equiv x\mathfrak{X}; \text{ therefore } \mathrm{X}' = \mathfrak{X} + x\mathfrak{X}'.$$

Dividing by x^m, the differential equation becomes

$$(a + b\mathfrak{X} + c\mathfrak{X}^2 + \text{----})(\mathfrak{X} + x\mathfrak{X}') = \mathrm{A} + \mathrm{B}\mathfrak{X} + \mathrm{C}\mathfrak{X}^2 + \text{----}.$$

From this is easily deduced

$$\frac{dx}{x} = \frac{a + b\mathfrak{X} + c\mathfrak{X}^2 + \text{-------}}{\mathrm{A} + (\mathrm{B} - a)\mathfrak{X} + (\mathrm{C} - b)\mathfrak{X}^2 + \text{-----}}.d\mathfrak{X}.$$

The integral of the left side is $\log x$. Therefore, if that on the right is directly integrable * to a function of $\mathfrak{X}$, say $\phi(\mathfrak{X}) \equiv \phi\left(\frac{\mathrm{X}}{x}\right)$; then the equation

$$\log x + \mathrm{C} = \phi\left(\frac{\mathrm{X}}{x}\right)$$

gives the desired integral relation between x and X.

A convenient shorthand symbol for such a homogeneous (x, X) – function of the m^{th} degree is $f(x^{m-r}, \mathrm{X}^r)$. The two such functions

* See Classified List, III. A. 19.

may be called $f(x^{m-r}, X^r)$ and $F(x^{m-r}, X^r)$. The given differential equation may then be written

$$X'f(x^{m-r}, X^r) = F(x^{m-r}, X^r).$$

Dividing $f(x^{m-r}, X^r)$ by x^m we obtain the same function of 1 and $\mathfrak{X}$ as $f(x^{m-r}, X^r)$ is of x and X. The quotient may, therefore, be written $f(1^{m-r}, \mathfrak{X}^r)$, and similarly that of $F(x^{m-r}, X^r)$ by x^m may be written $F(1^{m-r}, \mathfrak{X}^r)$.

The integral equation then appears as

$$\log x + C = \int \frac{d\mathfrak{X}}{\frac{F(1^{m-r}, \mathfrak{X}^r)}{f(1^{m-r}, \mathfrak{X}^r)} - \mathfrak{X}}.$$

204. Homogeneous Rational Functions.—The last form is a particular case of a more general one involving the first power of X′ only, and the **ratio only** of X to x. Call this ratio $\mathfrak{X}$ as in last article, so that as before $X' = \mathfrak{X} + x\mathfrak{X}'$. The present more general form of differential equation may be written

$$X' = f(\mathfrak{X}) = \mathfrak{X} + x\mathfrak{X}'$$

where $f(\ )$ indicates **any** form of function.

Therefore

$$\mathfrak{X}' = \frac{f(\mathfrak{X}) - \mathfrak{X}}{x}$$

or

$$\frac{dx}{x} = \frac{d\mathfrak{X}}{f(\mathfrak{X}) - \mathfrak{X}}.$$

The integral solution of this, namely,

$$\log x + C = \int \frac{d\mathfrak{X}}{f(\mathfrak{X}) - \mathfrak{X}}$$

gives x in terms of $\mathfrak{X} \equiv \frac{X}{x}$, and therefore gives X also in terms of x.

205. $X = xf(X')$.

A form of differential equation of cognate inverse character is that solved in § 200, namely,

$$X = xf(X')$$

or

$$\mathfrak{X} = f(X') \text{ where } \mathfrak{X} = \frac{X}{x}.$$

One solution of this is given in § 200. Otherwise, it may possibly be more easily solved algebraically so as to give X′ explicitly in terms of $\mathfrak{X}$. Let $f^{-1}(\ )$ denote the inverse of the function $f(\ )$. Then this algebraic solution would appear as

$$X' = f^{-1}(\mathfrak{X})$$

of which, according to last article, the integral solution is

$$\log x + C = \int \frac{d\mathfrak{X}}{f^{-1}(\mathfrak{X}) - \mathfrak{X}}.$$

206. $X' = (Ax + BX + C) \div (ax + bX + c)$.

A differential equation bearing a resemblance to that of § 203 is

$$(ax + bX + c)X' = Ax + BX + C.$$

If the two constants c and C did not appear, then by dividing by x, each (x,X) function would be converted into one involving the ratio only of the two variables. But c and C can be got rid of by shifting parallelly the axes of co-ordinates from which x and X are measured, which change does not affect X′. If $\bar{x}$ and $\overline{X}$ be the new co-ordinates, then it is easily shown that the axes must be shifted so as to make

$$\bar{x} = x - \frac{Bc - bC}{Ab - aB} \text{ and } \overline{X} = X - \frac{Ca - cA}{Ab - aB}.$$

Then, since $\frac{d\overline{X}}{d\bar{x}} = \frac{dX}{dx} = X'$, by dividing out by $\bar{x}$, there results an equation of the form dealt with in § 203.

207. Particular case, $B = -a$.

If in the equation of last article $B = -a$, then the two terms with the common factor a combine to make the complete increment of xX. Thus the equation then reduces to

$$(bX + c)dX + a(Xdx + xdX) = (Ax + C)dx$$

the integration of which gives

$$\tfrac{1}{2}bX^2 + cX + axX - \tfrac{1}{2}Ax^2 - Cx + K = 0$$

in which K stands for the integration constant.

208. $X' + X\mathfrak{X}' = \Xi$.

Let $\mathfrak{X}'$ and Ξ be given functions of x, of which $\mathfrak{X}'$ is a function whose integral by dx can be directly found, namely $\mathfrak{X}$. Then, if the differential equation between X, X′, and x be found to be

$$X' + X\mathfrak{X}' = \Xi;$$

this can be solved by the device of multiplying by what is called an "**integrating factor,**" which means a factor which converts both sides of the equation into directly integrable functions. The factor which does this in the present case is $e^{\mathfrak{X}}$, where $\mathfrak{X}$ is the integral of the given function $\mathfrak{X}'$ and e is the base of the natural logarithmic system. Since the x-gradient of $e^{\mathfrak{X}}$ is $e^{\mathfrak{X}}\mathfrak{X}'$, that of $\mathrm{X}e^{\mathfrak{X}}$ is $\mathrm{X}'e^{\mathfrak{X}} + \mathrm{X}\mathfrak{X}'e^{\mathfrak{X}}$. Therefore, multiplying both sides of the differential equation by $e^{\mathfrak{X}}$ and integrating, there is obtained the integral equation

$$\mathrm{X}e^{\mathfrak{X}} = \int \Xi e^{\mathfrak{X}} dx + \mathrm{C}.$$

This formula is of practical use only when $\Xi e^{\mathfrak{X}}$ is a function which can be integrated either directly or by help of some transformation.

209. $\mathrm{X}' + \mathrm{X}\mathfrak{X}' = \mathrm{X}^n\Xi$.

This process may be followed in solving the differential equation

$$\mathrm{X}' + \mathrm{X}\mathfrak{X}' = \mathrm{X}^n\Xi$$

because $\mathrm{X}^n\Xi$ is a function of x and may be inserted in place of Ξ in the above solution. Another solution, however, is obtained by dividing each side by X^n and multiplying by the integrating factor $(1-n)e^{(1-n)\mathfrak{X}}$. The x-gradient of $\mathrm{X}^r e^{s\mathfrak{X}}$ is

$$(r\mathrm{X}' + s\mathrm{X}\mathfrak{X}')\mathrm{X}^{r-1}e^{s\mathfrak{X}}$$

so that, taking $s = r$, the x-gradient of $\mathrm{X}^r e^{r\mathfrak{X}}$ is $(\mathrm{X}' + \mathrm{X}\mathfrak{X}')\mathrm{X}^{r-1} . re^{r\mathfrak{X}}$. The first factor here is identical with the left-hand side of the differential equation of this article when each side is divided by X^n if $r = 1 - n$. Therefore the integration gives

$$\mathrm{X}^{(1-n)}e^{(1-n)\mathfrak{X}} = (1-n)\int \Xi e^{(1-n)\mathfrak{X}} dx + \mathrm{C}.$$

Provided $\Xi e^{(1-n)\mathfrak{X}}$ be integrable, this formula will be of practical use.

210. $\mathrm{X} = x\mathrm{F}(\mathrm{X}') + f(\mathrm{X}')$.

The following equation is generalised from that of § 202 by inserting the general function $\mathrm{F}(\mathrm{X}')$ of the x-gradient of X, in place of the special simple function X′.

$$\mathrm{X} = x\mathrm{F}(\mathrm{X}') + f(\mathrm{X}').$$

Taking the x-gradient of each side,

$$\mathrm{X}' = \mathrm{F}(\mathrm{X}') + \{x\mathrm{F}'(\mathrm{X}') + f'(\mathrm{X}')\}\frac{d\mathrm{X}'}{dx}.$$

Transposing this,

$$\frac{dx}{d\mathrm{X}'} - x\frac{\mathrm{F}'(\mathrm{X}')}{\mathrm{X}' - \mathrm{F}(\mathrm{X}')} = \frac{f'(\mathrm{X}')}{\mathrm{X}' - \mathrm{F}(\mathrm{X}')}.$$

This is of the same form as that of § 208 with the substitutions

$$
\begin{array}{rl}
x & \text{in place of } X \\
X' & \text{,, ,, ,, } x \\
-\dfrac{F'(X')}{X'-F(X')} & \text{,, ,, ,, } \mathfrak{X}'
\end{array}
$$

and

$$
\frac{f'(X')}{X'-F(X')} \text{ ,, ,, ,, } \Xi.
$$

Therefore, if we use the shorthand symbol

$$
\mathfrak{X} \text{ for } -\int\frac{F'(X')}{X'-F(X')}dX'
$$

so that $\mathfrak{X}$ is a function of X′ only; and further use

$$
\Xi \text{ for } \frac{f'(X')}{X'-F(X')},
$$

another function of X′ only; the integration gives

$$
xe^{\mathfrak{X}} = \int \Xi e^{\mathfrak{X}} dX' + C;
$$

which, if $\Xi e^{\mathfrak{X}}$ be directly integrable by X′, gives an algebraic equation between x and X′. Combining this with the original equation as simultaneous, the algebraic elimination of X′ gives the desired integral equation involving only x and X.

211. General Equation of 1st Order of any Degree.—The preceding differential equations contain X′ in the first power only. The general equation of the first order and of any degree may be expressed thus:—

$$
X'^n + \mathfrak{X}_{n-1}X'^{n-1} + \mathfrak{X}_{n-2}X'^{n-2} + \cdots + \mathfrak{X}_1 X' + \mathfrak{X}_0 = 0
$$

where $\mathfrak{X}_{n-1}, \mathfrak{X}_{n-2}, \cdots \mathfrak{X}_0$ are n different functions involving both x and X, and n is the degree of the equation. If possible this should first be solved for X′ algebraically in terms of x and X. There will be n solutions giving n values of X′ which may be symbolised by $X'_n, X'_{n-1}, \cdots X'_2, X'_1$, each of these values of X′ being expressed in terms of x and X. This reduces the above equation of the n^{th} degree to an equivalent series of n linear or first-degree equations; for instance the first of this series is

$$
X' - X'_n = 0
$$

where X'_n is a function, supposed now to be known, of x and X. Integrate each such linear equation if possible, by one of the methods already given. Let the integral solutions be here symbolised by

$$\phi_n(x, X, C) = 0\,;\ \phi_{n-1}(x, X, C) = 0\,;\ \text{etc., etc.}$$

Then the general solution, that is, the equation which includes all these various solutions, is either

$$\phi_n(x, X, C) \cdot \phi_{n-1}(x, X, C) \text{-----} \phi_2(x, X, C) \cdot \phi_1(x, X, C) = 0$$

or some other algebraically legitimate combination of these solutions.

212. Quadratic Equation of First Order.—Applying the result of last article to the equation of the second degree, namely,

$$X'^2 + X'\mathfrak{X} + \Xi = 0\,.$$

The algebraic solution of this for X' in terms of $\mathfrak{X}$ and Ξ is

$$X' = -\frac{\mathfrak{X}}{2} \pm \frac{1}{2}\sqrt{\mathfrak{X}^2 - 4\Xi}\,.$$

The two integral solutions are, therefore,

$$\left.\begin{aligned} X + \tfrac{1}{2}\int \mathfrak{X}dx - \tfrac{1}{2}\int \sqrt{\mathfrak{X}^2 - 4\Xi}\,dx + C = 0 \\ \text{and} \qquad X + \tfrac{1}{2}\int \mathfrak{X}dx + \tfrac{1}{2}\int \sqrt{\mathfrak{X}^2 - 4\Xi}\,dx + C = 0\,. \end{aligned}\right\}$$

As an easy example, take the differential equation

$$X'^2 + X'\sin x - \frac{\cos^2 x}{4} = 0\,.$$

Therefore, $\sqrt{\mathfrak{X}^2 - 4\Xi} = \sqrt{\sin^2 x + \cos^2 x} = 1$; and the two integral solutions are

$$\left.\begin{aligned} X = \tfrac{1}{2}(\cos x - x) + C \\ \text{and} \qquad X = \tfrac{1}{2}(\cos x + x) + C\,. \end{aligned}\right\}$$

213. Equation of Second Order with One Variable Absent.—In differential equations of the second order with only one independent variable, there may appear powers, trigonometrical, or any other kind of functions of all the four quantities

$$X'',\ X',\ X \text{ and } x.$$

Now X″ may be expressed in terms of X′ and either x or X by means of the substitutions

$$X'' \equiv \frac{dX'}{dx} \qquad (a)$$

$$\equiv X'\frac{dX'}{dX} \qquad (b).$$

Therefore, if in any second-order differential equation X does not appear, it may be transformed by help of the substitution (a) so as to make X″ also disappear, leaving only

$$\frac{dX'}{dx},\ X' \text{ and } x.$$

This is an equation of the **first order** between X′ and x, and may be solved by methods already explained so as to give X′ as a function of x $\left(i.e., \text{ so as to eliminate } \frac{dX'}{dx}\right)$. This again is a first-order equation between X and x, and by a second similar solution we may pass to the integral equation between X and x.

On the other hand, if x does not appear in any second-order equation, it may be reduced by the substitution (b), so that it will involve only

$$\frac{dX'}{dX},\ X' \text{ and } X.$$

This is an equation of the **first order** again between X′ and X, whose solution gives an equation between X′ and X not involving $\frac{dX'}{dX}$; and from this again by a second integration, the desired integral relation between X and x may result.*

214. Second Order Linear Equation.—The linear, or 1st degree, equation of the second order appears in a very general form as

$$X'' + X'f(x) + XF(x) = \phi(x),$$

where f, F and ϕ are **any** forms of function.

Provided this equation can be solved when $\phi(x) = 0$; then also, when $\phi(x)$ is **any** function, it may be reduced to an equation of the 1st order. Thus, let Ξ be a function of x which would be a solution if $\phi(x)$ were zero; that is, let

$$\Xi'' + \Xi'f(x) + \Xi F(x) = 0.$$

* See Reference List, XI. C. 5.

Give the name $\mathfrak{X}$ to the ratio of X, the true solution of the given equation, to Ξ; that is, let

$$X = \mathfrak{X}\Xi ;$$

Then

$$X' = \mathfrak{X}'\Xi + \mathfrak{X}\Xi'$$

and

$$X'' = \mathfrak{X}''\Xi + 2\mathfrak{X}'\Xi' + \mathfrak{X}\Xi''.$$

Therefore, inserting these substitutions in the original equation, it becomes

$$\mathfrak{X}''\Xi + \mathfrak{X}'\{2\Xi' + \Xi f(x)\} + \mathfrak{X}\{\Xi'' + \Xi' f(x) + \Xi F(x)\} = \phi(x) .$$

But the bracketed factor of the third term is zero; so that the transformed equation becomes

$$\mathfrak{X}''\Xi + \mathfrak{X}'\{2\Xi' + \Xi f(x)\} = \phi(x) .$$

The supposition is that Ξ has been found; from which Ξ' can also be found in terms of x. This last form of the equation therefore contains only known functions of x besides $\mathfrak{X}''$ and $\mathfrak{X}'$. Now $\mathfrak{X}''$ is the first x-gradient of $\mathfrak{X}'$; and this is therefore a 1st order linear equation as between $\mathfrak{X}'$ and x. Thus, if any of the already explained, or any other, method of integrating 1st order linear equations be applicable, then $\mathfrak{X}'$ can be found as an explicit or implicit function of x, thus giving another 1st order equation between $\mathfrak{X}$ and x. By a second integration by one of these same methods, $\mathfrak{X}$ may then be found as a function of x; and finally $X = \mathfrak{X}\Xi$ can be obtained as the desired solution.*

215. $X'' + aX' + bX = 0$.

The equation determining Ξ in § 214 is soluble or not according to the particular forms of the functions $f(\ )$ and $F(\ )$: at any rate no reduction of the equation has yet been discovered showing, independently of the forms of $f(\ )$ and $F(\ )$, how it may be solved.

One simple case is that in which these functions are both **constants.** Let $f(x) \equiv a$ and $F(x) \equiv b$, a and b being both constants. The equation is then, using X instead of Ξ,

$$X'' + aX' + bX = 0 .$$

Using the substitution (b) of § 213, this becomes

$$\left(a + \frac{dX'}{dX}\right)X' + bX = 0.$$

This may be written

* See Reference List, XI. C. 7.

$$\frac{dX'}{dX} = -\left(b\frac{X}{X'} + a\right)$$

which is soluble by the method of § 204; or it may be written

$$X' = -\frac{bX}{a + \dfrac{dX'}{dX}}$$

which is soluble by § 200.

By either method the solution is obtained which is printed in the *Classified Reference List*, at XI. C. 3.

216. $X'' + aX' + bX = \phi(x)$.

In this simple case of $f(x) = a$ and $F(x) = b$, the more general equation of § 214 becomes

$$X'' + aX' + bX = \phi(x);$$

and its reduced form, when divided out by Ξ, becomes

$$\mathfrak{X}'' + \mathfrak{X}'\left\{2\frac{\Xi'}{\Xi} + a\right\} = \frac{\phi(x)}{\Xi}.$$

By § 215 both Ξ and Ξ' are known functions of x; and, therefore, this equation is of the form given in § 208, and can be integrated so as to give $\mathfrak{X}'$, provided the function $\left\{2\frac{\Xi'}{\Xi} + a\right\}$ can be integrated by dx. This is integrable because it is found that

$$2\frac{\Xi'}{\Xi} = -a + \sqrt{a^2 - 4b}\left\{\frac{1}{1 + \frac{B}{A}e^{-x\sqrt{a^2-4b}}} - \frac{1}{1 + \frac{A}{B}e^{x\sqrt{a^2-4b}}}\right\} \quad \text{when } a^2 > 4b$$

$$= -a - \sqrt{4b - a^2}\tan\left\{\frac{x}{2}\sqrt{4b - a^2} + C'\right\} \qquad \text{when } a^2 < 4b.*$$

217. $X^{(n)} = f(x)$.

If the n^{th} x-gradient of X be called $X^{(n)}$, and the process of repeating the integration of a function n times be symbolised by $\int^{(n)}$; then, if the differential equation of the n^{th} order be

$$X^{(n)} = f(x)$$

it has already been shown in § 154 that the integral equation between X and x is

$$X = \int^{(n)} f(x)dx^n + C_{n-1}x^{n-1} + C_{n-2}x^{n-2} + \cdots + C_1x + C_0. \dagger$$

* See Reference List, XI. C. 6 and 3. † See Reference List, XI. D 3.

218. $X^{(n)}=f(X):X^{(n)}=kX$.

If the equation of the n^{th} order be

$$X^{(n)}=f(X)$$

it is integrable only in particular cases. Thus in § 153 is given the case

$$X^{(n)}=kX$$

where k is any number + or −. Let b be any number, and let T be the "modulus" of the system of logarithms of which the base is b. Take $\beta=Tk^{1/n}$. Then a solution of the above equation is

$$X=b^{\beta x} \text{ or } \log_b X=\beta x .$$

If b be taken equal to e, the base of natural logarithms, then $T=1$, and the solution is

$$\log_e X=k^{1/n}x .$$

If decimal logarithms be used, or $b=10$; then $T=\cdot 434$, and

$$\log_{10}X=\cdot 434k^{1/n}x .$$

Again in § 153 it is shown that if n be an even number, and if

$$X^{(n)}=(-1)^{n/2}kX$$

then an integral solution is

$$X=A\sin k^{1/n}x+B\cos k^{1/n}x$$

where A and B are constants of integration.*

219. $X''=f(X)$.

When $n=2$, this equation becomes

$$X''=f(X).$$

A general rule independent of the form of $f(\ )$ has been found for dealing with this second-order equation. Multiply each side by $2X'$. Then since $2X'X''$ is the x-gradient of X'^2, and since $X'dx=dX$, there results

$$X'=\{2\int f(X)dX+A\}^{\frac{1}{2}};$$

from which, by another integration,

$$x+B=\int\frac{dX}{\{2\int f(X)dX+A\}^{\frac{1}{2}}}.\dagger$$

* The $1/n^{th}$ root of k has n values. The insertion of these gives the n integration-constants of this n-th order equation.

† See Reference List, XI. C. 5.

As examples, the results of §§ 148 and 149 may be reproduced; but these are included in the more general formulas of last article, § 218.

220. $X^{(n)} = f(X^{(n-1)})$.

If $X^{(n)}$ be found as a function of $X^{(n-1)}$; then, since the x-gradient of $X^{(n-1)}$ is $X^{(n)}$, if we call $X^{(n-1)}$ by the name $\mathfrak{X}$, the equation may be written

$$\mathfrak{X}' = f(\mathfrak{X}),$$

the integration of which by dx gives

$$x + C = \int \frac{d\mathfrak{X}}{f(\mathfrak{X})},$$

that is, gives $X^{(n-1)}$ as a function of x, a case which has been already dealt with in § 217.*

221. If $X^{(n)}$ be found as a function of $X^{(n-2)}$; then, calling $X^{(n-2)}$ by the name $\mathfrak{X}$, we have $X^{(n)} = \mathfrak{X}''$, and the equation becomes

$$\mathfrak{X}'' = f(\mathfrak{X}),$$

the integration of which, by § 219, gives $X^{(n-2)}$ as a function of x, and this reduces the integration to the case of § 217.†

More general forms of equation, to which these last substitutions are equally applicable, are given in the Section XI. D. of the *Reference Tables.*

222. If $f(\ )$ and $\phi(\ )$ be two functions of any form whatever, and if

$$X = f\left(x + \frac{y}{c}\right) + \phi\left(x - \frac{y}{c}\right),$$

the second gradients of X with respect to x and y may easily be found to be

$$\frac{d^2X}{dx^2} = f''\left(x + \frac{y}{c}\right) + \phi''\left(x - \frac{y}{c}\right) \text{ and}$$

$$\frac{d^2X}{dy^2} = \frac{1}{c^2}f''\left(x + \frac{y}{c}\right) + \frac{1}{c^2}\phi''\left(x - \frac{y}{c}\right).$$

Therefore, if the second-order differential equation

$$\frac{d^2X}{dx^2} = c^2\frac{dX^2}{dy^2}$$

be known to be true, its GENERAL integral solution is X = the above form, and the particular forms of the functions $f(\ )$ and $\phi(\ )$ must be discovered from the limiting conditions of the particular concrete case.

* See Reference List, XI. D. 1. † See Reference List, XI. D. 2.

APPENDICES.

APPENDIX A.—TIME-RATES.

(End of Chap. II., p. 28.)

THE Differential and Integral Calculus was first studied as an exact method of analysis of physical phenomena occurring in *time*, chiefly kinetic phenomena. The changes of observed physical condition occur from instant to instant, and an "*instant*," or small lapse of time, was taken as the common measure by which to compare *simultaneously* occurring changes of various kinds. Thus *time* was taken as the base ordinate of the diagrams which graphically describe such changes. The flow of a fluid along a channel is the simplest possible illustration of such change or progress, and all phenomena were thought of as developing in the flow, or *flux*, of time, the universal basic *increment* being a small *flux* of time. Thus the early name given to the then new method of analysis was "Fluxions." Unless it were otherwise specified, x' or $\dot{x}$ was understood to mean the *time-rate* at which x increased, and the relative rates of increase of various kinds of quantities were always obtained by comparing their respective *simultaneous time-rates* of progress or development. So long as investigation deals with things which "take time" to develop or change in magnitude, it will be found that this original method corresponds with our innate and almost ineradicable mental habit. The *corresponding* increments of such things we can hardly avoid thinking of as those which are developed *in the same time.*

APPENDIX B.—ENERGY-FLUX.

(End of § 68, p. 33, Chap. III.)

Energy manifests itself to our means of observation and measurement in various forms, such as kinetic, electric, thermal, luminous (light), sonorous (sound), gravity potential, electro-magnetic potential, radiant, etc. These are reciprocally convertible, and are, therefore, all measurable in like "physical dimensions," namely, MV^2 or ML^2T^{-2}. As energy is, or is believed to be, indestructible, the variations it is subject to are (1) change of form; (2) transference from one mass to another mass; and (3) transference from one place to another place.

The *time-rate* of transference of energy is *horse-power*; a special unit time-rate being adopted as unit horse-power. Unfortunately, many unit time-rates of energy-variation are in use; but they are all, of course, of the same kind, namely, horse-power. In terms of mass and velocity the measure of energy is $E = \frac{1}{2}MV^2$. The *time-gradient* of E in a *constant mass M*, due to variation of velocity V in that mass, is, therefore, $\frac{dE}{dt} = VM\frac{dV}{dt} = V \cdot F$, where F is the time-acceleration of momentum, or the *force* active in the transference of energy. It may also sometimes be usefully thought of as the product of the momentum and the velocity acceleration.

The *space-rate* or *line-gradient* of E with M constant is

$$\frac{dE}{dl} = MV\frac{dV}{dl} = MV\frac{dV}{dt} \cdot \frac{dt}{dl} = M\frac{dV}{dt} = F, \text{ because } V = \frac{dl}{dt}.$$

Thus the two important energy-gradients are the time-gradient or horse-power, and the line-gradient or active dynamic force; and the former equals the latter multiplied by the velocity.

It is also interesting to consider the time-gradient of E with both M and V varying together. When a mass receives new energy from without, it absorbs it usually (and perhaps always) at its surface, and the new energy spreads through the mass with more or less rapidity or slowness. New impulses of kinetic energy by impact or pressure of other masses always enter and penetrate the accelerated mass in this way. In such case the time-gradient, or horse-power generating kinetic energy in the mass, is

$$\frac{dE}{dt} = VF + \frac{1}{2}V^2\frac{dM}{dt} = V\left(F + \frac{1}{2}V\frac{dM}{dt}\right) = \frac{1}{2}\left\{(MV)\frac{dV}{dt} + V\frac{d(MV)}{dt}\right\}.$$

Here $\frac{dM}{dt}$ is the time-rate at which new mass is affected by the kinetic energy, and (MV) is the whole momentum acquired at any instant.

Appendix C.—Moments of Inertia and Bending Moments.

(End of § 73, p. 37, Chap. III.)

The integral $\int bh^2dh$ over the whole section is called the "Moment of Inertia" of the section. For an I-section with equal flanges and web of uniform thickness, if B be the flange width, $(1-\beta)$B the web thickness, H the whole depth, and ηH the depth inside the flanges; then the

$$\mathrm{I} = \text{Moment of Inertia} = \frac{\mathrm{BH}^3}{12}(1-\beta\eta^3), \text{ and the}$$

$$\mathrm{M} = \text{Stress Bending Moment} = k\frac{\mathrm{BH}^2}{6}(1-\beta\eta^3).$$

The sectional area is $\mathrm{A} = \mathrm{BH}(1-\beta\eta)$, and therefore the stress bending-moment strength per square inch of section

$$= \frac{\mathrm{M}}{\mathrm{A}} = k\frac{\mathrm{H}}{6}\ \frac{1-\beta\eta^3}{1-\beta\eta}.$$

Girder-sections are mostly made up of rectangular parts, and the repeated application of the method here given is usually sufficient for the calculation of their moment strength. In making such calculations, free use should be made of *negative* rectangular areas as parts of the section.

Appendix D.—Elimination of Small Remainders.

(End of Chap. III., p. 45.)

In previous examples given, the device of taking the point x,X at the middle of δx and assuming this to correspond also to the middle of δX, that is, assuming linear proportionality between δX and δx, has resulted in the exact elimination of all small remainders. The case of $\frac{\mathrm{X}}{x}$ is a useful illustration of the fact that such exact elimination does not always result from this device. In this case the result is

$$\delta\left(\frac{\mathrm{X}}{x}\right) = \frac{\mathrm{X}+\frac{\delta\mathrm{X}}{2}}{x+\frac{\delta x}{2}} - \frac{\mathrm{X}-\frac{\delta\mathrm{X}}{2}}{x-\frac{\delta x}{2}} = \frac{x\mathrm{X}'-\mathrm{X}}{x^2-\frac{1}{4}\delta x^2}\cdot\delta x,$$

the small quantity $\frac{1}{4}\delta x^2$ not being eliminated and only disappearing "in the limit."

The student should satisfy himself that the same result appears from the geometrical method followed in the text, with fig. 18 modified so as to put x and X in the centres of δx and δX.

Appendix E.—Indicator Diagrams.

(End of § 111, p. 65, Chap. V.)

An "indicator diagram" is any instrumental graphic record of a varying quantity. The law $pv^n = k$, a constant, applies approximately to very many such records when the variation is not of an elastic vibratory kind. The p and the v instrumentally observed and recorded may not be the totals measured from absolute zero of the quantities thus symbolised. For instance, p may be pressure measured from atmospheric standard, or it may be temperature

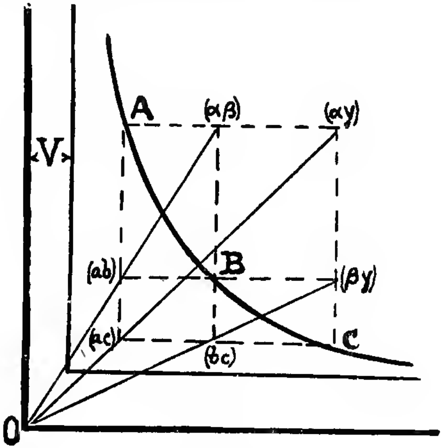

Fig. 30.

measured from the freezing-point of water. Again, v may be the volume swept by a piston from the beginning of its stroke, while the real volume of expanding steam or gas is v plus the "clearance" volume. The law connecting the absolute values of p and v may be much simpler than appears at first sight from the indicator record. To determine whether it is so, the axes of the absolute zeros must be discovered, making the ordinates measured from these zeros $(p + \text{P})$ and $(v + \text{V})$. The constants P and V may be found by analysis of the recorded curve.

If it be suspected that the curve is hyperbolic with $\text{V} = \text{O}$, then P may be found by measuring from the record p and v at two points 1 and 2; thus,

$$\text{P} = (p_1 v_1 - p_2 v_2) \div (v_2 - v_1).$$

If the same value of P be found from several such pairs of points, then the curve is truly hyperbolic. Similarly, V can be found on the assumption that $P = O$. Also by measurements at three points, both P and V along with k can be calculated. Fig. 30 shows the very simple graphic construction for finding the origin of the hyperbolic axes. From three points ABC on the curve horizontals and verticals are drawn, giving six intersections, marked as shown on fig. 30; namely, $(\alpha\beta)$ the intersection of the horizontal through A with the vertical through B, (ab) that of the vertical at A with the horizontal at B, etc., etc. The pairs of points are joined by straight lines: $(\alpha\beta)$ with (ab); $(\beta\gamma)$ with (bc); $(\alpha\gamma)$ with (ac). Any two of these three lines will intersect in the hyperbolic origin O. There are three such intersections, and their coincidence is a useful check upon the accuracy of the draughtsmanship. But to test whether the curve be truly hyperbolic, more than three curve-points must be used; they must all give the same origin O.

To find the index n in the formula $pv^n = k$ to fit the curve of any given indicator card, there are four methods. The first, most commonly used, is the *logarithmic* method $n = \dfrac{\log v_2 - \log v_1}{\log p_1 - \log p_2}$. The second is the *differential* method, which gives $n = -\dfrac{v}{p}\dfrac{dp}{dv}$, the slope of the curve, or $\dfrac{dp}{dv}$, being measured directly from the paper. The third method utilises the variation of pv; it is $n = 1 - \dfrac{1}{p}\dfrac{d(pv)}{dv}$. In the hyperbola $\dfrac{d(pv)}{dv} = 0$; and in any other curve its deviation from zero gives a good measure of the index n. The fourth method may be called the *integral* method. Since the area under the curve $W = \dfrac{p_1v_1 - p_2v_2}{n-1}$, therefore $n = 1 + \dfrac{p_1v_1 - p_2v_2}{W}$. The area W can be measured from the diagram by a planimeter or otherwise, and this method has the great merit of deducing the result, not from isolated points on the curve alone, but from the curve *as a whole* in its stretch between two distant points 1 and 2.

The generality of the formula, and its power to represent accurately recorded physical results, is very greatly extended by shifting the axes by adding constants P and V, the formula being then $(p + P)(v + V)^n = k$. The measurement of p and v at *four* points of the curve, and the planimeter integration of the area

$W = \int p dv$ between these points, are sufficient to determine the four constants k, n, P, and V. The three areas under the three stretches of the curve between the four points being called W_{12}, W_{23}, and W_{34}, the constants n, P, and V are obtained by elimination from the three linear equations—

$$W_{12}(n-1) = p_1 v_1 - p_2 v_2 + (p_1 - p_2)V - Pn(v_2 - v_1),$$
$$W_{23}(n-1) = p_2 v_2 - p_3 v_3 + (p_2 - p_3)V - Pn(v_3 - v_2),$$
$$W_{34}(n-1) = p_3 v_3 - p_4 v_4 + (p_3 - p_4)V - Pn(v_4 - v_3).$$

These are solved for $(n-1)$, V, and Pn, from which n, V, and P are obtained. Then the fourth constant is obtained by the primitive equation

$$k = (p + P)(v + V)^n,$$

taking any pair of values p and v.

With the four constants thus found, the formula will give a curve which will pass correctly through the four points, and will give also correctly the area under the stretch of curve between each pair of points.

When $n = 0$, the curve is a horizontal straight line, giving constant p. When n is negative it slopes upwards, p increasing with v.

In this extended form this formula is well suited to represent all systematic expansions of gases and vapours under various thermal conditions, the stress-strain relations of elastic and inelastic solids, the strain-time diagram of the slow deformation of plastic substances, and, under certain conditions, also the leakage of high-pressure water, gas, vapour, or electric charge from storage vessels, batteries, condensers, etc., and the laws connecting velocity with wind-pressure and with viscous fluid resistance to flow.

Appendix F.—Recurrent Harmonic and Exponential Functions.

(End of § 153, p. 89, Chap. VII.)

The combination of these two cases gives the following interesting much more general result.

Let A, B, C, b, β, a, p, q, k be constants, and let the primitive function be

$$X = C + b^{\beta x + a}\{A \sin(px + q) + B \cos(px + k)\}.$$

Let $\rho^2 = (\beta \log b)^2 + p^2$ and $\theta = \tan^{-1} \dfrac{p}{\beta \log b}$,

so that $\sin \theta = \dfrac{p}{\rho}$ and $\cos \theta = \dfrac{\beta \log b}{\rho}$.

Then direct differentiation and the ordinary trigonometric combinations

$$\sin \phi \cos \theta + \cos \phi \sin \theta = \sin (\phi + \theta)$$
$$\cos \phi \cos \theta - \sin \phi \sin \theta = \cos (\phi + \theta),$$

give

$$X' = \rho b^{\beta x + \alpha}\{A \sin (px + q + \theta) + B \cos (px + k + \theta)\}$$
$$X'' = \rho^2 b^{\beta x + \alpha}\{A \sin (px + q + 2\theta) + B \cos (px + k + 2\theta)\}$$
$$X^{(n)} = \rho^n b^{\beta x + \alpha}\{A \sin (px + q + n\theta) + B \cos (px + k + n\theta)\}.$$

The only restriction among the constants is that the constant p is the same in the two harmonic functions. If x measures *time*, the equality of the two p's means that the two superimposed vibrations have equal *periods* or frequencies. Their difference in *phase* is $(k - q)$, and this is unrestricted. In all the successive x- or time-gradients, the phase-difference, as well as the frequency, remains the same in the two superimposed harmonics. In each gradient the frequency is the same as in the primitive. The phase-difference between each gradient and the next lower gradient is $\theta = \tan^{-1} \dfrac{p}{\beta \log b}$. The factor $b^{\beta x + \alpha}$ represents the damping down of the vibration, as its energy is gradually reduced by viscous or similar dissipative resistances. The amplitudes of the successively higher gradients are successively less in the ratio ρ. This ratio ρ depends equally upon the frequency $\left(\text{the period} = \dfrac{2\pi}{p}\right)$ and upon the vigour of the damping coefficient b^{β} or $\beta \log b$.

This proposition can evidently be extended to the superposition of any number of harmonic vibrations of different amplitudes A, B, etc., and of different phases, so long as the frequency is the same in all and so long as the same damping coefficient applies to all. It is of the highest practical importance in modern electrical industry. See also Appendix Q, page 184.

APPENDIX G.—SUCCESSIVE REDUCTION FORMULA.

(End of § 156, p. 91, Chap. VII.)

The "reduction formula" of § 126 may be applied repeatedly. Such application is represented by the general formula below. The repeated application in concrete special cases is, however, simpler to appreciate than is the general result.

Let X and Ξ be any functions of x.
Let D be the m^{th} x-gradient of $X = X^{(m)}$.

Let I be the m^{th} integral of $\Xi = \int^{(m)} \Xi dx^m$.

Then $D' = X^{(m+1)}$ and $\int I dx = \int^{(m+1)} \Xi dx^{m+1}$.

Now $\int DI dx = D \int I dx - \int D' \left\{ \int I dx \right\} dx,$

$$\text{or} \int X^{(m)} \int^{(m)} \Xi dx^{m+1} = X^{(m)} \int^{(m+1)} \Xi dx^{m+1} - \int X^{(m+1)} \int^{(m+1)} \Xi dx^{m+2}.$$

Applying this repeatedly,

$$\int X \Xi dx = X \int \Xi dx - \int X' \int \Xi dx^2 = X \int \Xi dx - X' \int^{(2)} \Xi dx^2 + \int X'' \int^{(2)} \Xi dx^3$$

$$= X \int \Xi dx - X' \int^{(2)} \Xi dx^2 + X'' \int^{(3)} \Xi dx^3 - X''' \int^{(4)} \Xi dx^4 + \cdots$$

$$\pm X^{(n-1)} \int^{(n)} \Xi dx^n \mp \int X^{(n)} \int^{(n)} \Xi dx^{(n+1)}. *$$

By this means the function X and its derivatives are brought outside the sign of integration except in the last term. By carrying the series to the proper number (n) of terms, in the last term $X^{(n)}$, or the n^{th} x-gradient of X, may be reduced to 1 or to a constant, or to some simple function of x which combines with the n^{th} integral of Ξ to form an integrable function. This last term may also be transformed to

$$\int X^{(n-1)} \left\{ \int^{(n)} \Xi dx^n \right\} dX,$$

in which form it may possibly be more easily integrable.

* See Reference List, I. 8.

Appendix H.—Economic Proportions of I-Sections.

(End of § 185, p. 110, Chap. IX.)

The economic values of $\frac{h}{\mathrm{H}}$ for given w, according to the equation in the text, are as follows:—

w	0	·018	·049	·101	·185	·315	·513	·815	1
$\frac{h}{\mathrm{H}}$	1	·9	·85	·8	·75	·7	·65	·6	·5773

In two articles in the *Builders' Journal and Architectural Engineer* of 15th August 1906, and in *The Engineer* of 9th November 1906, the author has shown that the proper proportioning of I-sections depends on the consideration of the maximum tensive, compressive, and shear stresses on *oblique* sections, which vary very differently from those on normal sections. If the web be made too thin, the maximum oblique tensive and compressive stresses at junction of web and flange are greater than at the outside surfaces of the flanges. With proper proportioning of the web and flange thicknesses to the flange width and the total depth, the maximum oblique shear stress may be made uniform throughout the depth of the web, while at the same time the tensive and compressive stresses on oblique sections inside the flanges are made equal to those at the outside surfaces of the flanges.

Using δ for $\frac{h}{\mathrm{H}}$ and $(1-\beta)$ for the above w, it is shown that uniformity of shear stress over the web is obtained by making

$$\frac{4(1-\beta)}{2-(1+\beta)\delta^2}=\left(\frac{\mathrm{M}'}{\mathrm{M}}\mathrm{H}-\mathrm{H}'\right)^2$$

where M′ and H′ mean the rates at which the bending moment and the section depth vary per inch along the span, M′ being, as is well known, equal to the total shear load on the section. Equality between the two most dangerous tensive and compressive stresses, at extremities of web and outside flanges, is obtained by making

$$\frac{16(1-\beta)^2}{(1+\delta)^2(1-\delta)}=\left(\frac{\mathrm{M}'}{\mathrm{M}}\mathrm{H}-\mathrm{H}'\right)^2.$$

The combination of these two equations determines a relation between β and δ independent of M, M′, H, and H′. This relation is

$$(1+\delta)^2(1-\delta)=4(1-\beta)\{2-(1+\beta)\delta^2\},$$

a relation which is very closely approximated to by the simpler equation

$$\beta=\cdot 805+\tfrac{1}{2}(\delta-\cdot 72)^2+(\delta-\cdot 72)^3.$$

A minimum value of $\beta=\cdot 805$ is reached at $\delta=$ about $\cdot 7$. It is shown that for the standard case of a uniformly distributed load on a beam freely supported at its two ends, these adjustments lead with very close approximation to the proportion

$$\delta=\cdot 72\frac{M_c}{T_m L^2 B}$$

where M_c is the central bending moment, L the span, B the flange width, and T_m the outside normal stress on flanges.

Appendix I.—Economic Design of Turbines.

(End of Chap. IX., p. 117.)

In *The Engineer* of 27th May, 10th June, and 17th June 1904, will be found a series of interesting calculations by the author on "Dynamic and Commercial Economy in Turbines." It is there shown that for greatest dynamic efficiency the angles between the periphery of the rotating wheel and the blades should be equal at entrance and at exit, that the tangent of this angle should be double the tangent of the peripheral angle of the fixed entrance guide-blades, and that the peripheral velocity of the wheel should be half the peripheral component of the water entrance-velocity. The angle of the rotating blades being called β, that of the fixed guides γ, and the water-velocity through these guides w, and the wheel-velocity b, these conditions give

$$2\tan\beta=\tan\gamma \text{ and } b=\frac{w}{2}\cos\gamma.$$

The linear velocity with which the water is fed into, and discharged from, the wheel is then $w\sin\gamma$.

Calling the dynamic efficiency ϵ, the power in ft.-lbs. per second delivered by the water to the wheel H, and the water-covered gate-area A, we find for water-turbines

$$\epsilon = \cos^2 \gamma \quad \text{and} \quad H = \cdot 97 A w^3 \sin \gamma \cos^2 \gamma.$$

The smaller γ is made the nearer does ϵ approach unity; but the γ-gradient of the power developed is

$$H' = \cdot 97 A w^3 \cos \gamma (1 - 3 \sin^2 \gamma),$$

and this gives maximum power at

$$\sin^2 \gamma = \tfrac{1}{3} \text{ or } \gamma = 35^\circ\ 16' \text{ with } \therefore\ \beta = 54^\circ\ 44' \text{ and } \epsilon = \frac{2}{3}.$$

This would be the best angle for the design if, for a prescribed size of wheel (measured here by A) and prescribed entrance-velocity w, the chief aim was to obtain maximum horse-power without consideration of water-consumption. This, however, does not mean maximum commercial economy.

The power "in the water" consumed is $\frac{H}{\epsilon}$. Suppose that the cost of each extra unit of water-power consumed is p, while the value of each extra unit of power developed (H) is h. Then the "net revenue" or "working profit" obtained from the use of the turbine is shown to be

$$R = \cdot 97 A h w^3 \sin \gamma \left(\cos^2 \gamma - \frac{p}{h} \right) - C$$

where C represents initial costs incurred whatever power be taken from the wheel.

The γ-gradient of R is

$$R' = \cdot 97 A h w^3 \cos \gamma \left(1 - 3 \sin^2 \gamma - \frac{p}{h} \right).$$

The maximum net revenue R for given size A and given water-velocity w is thus obtained with guide-blade angle giving

$$\sin^2 \gamma = \tfrac{1}{3} \left(1 - \frac{p}{h} \right) \text{ and } \epsilon = \frac{2}{3} + \frac{1}{3} \frac{p}{h},$$

which γ is less than for maximum power ($H' = 0$) in that $\frac{p}{h}$ is here subtracted from unity. Here $\frac{p}{h}$ is the ratio of cost of extra unit of water-power consumed to value of extra unit of power utilised. With γ so adjusted the maximum revenue is

$$R_{\max} = \cdot 373 A w^3 h \left(1 - \frac{p}{h}\right)^{\frac{3}{2}} - C.$$

For different ratios of p to h the following are the results :—

$\frac{p}{h}$ . . .	0	·1	·2	·3	·4	·5	·6	·7	·8	·9
γ degrees .	35·3	33·2	31·1	28·9	26·5	24·1	21·4	18·4	15·0	10·5
$\cdot 373\left(1 - \frac{p}{h}\right)^{\frac{3}{2}}$	·373	·319	·266	·218	·173	·132	·094	·061	·033	·012
ϵ . . .	·67	·70	·73	·76	·80	·83	·87	·90	·93	·97

As the relative cost of the water-power consumed goes up, dynamic efficiency becomes of greater commercial importance and the power extracted from the wheel with greatest commercial economy decreases.

Here there is no question of varying the size of the turbine. The problem is confined to finding the best power to extract from a given size of wheel under given conditions.

If the problem be how best to obtain a prescribed horse-power, the solution is quite different. The relative costs of providing a larger or smaller turbine have to be compared with the relative costs of working these at greater or smaller dynamic efficiencies. In this problem the value of the horse-power utilised is not involved. The prime cost of the installation must be reduced to a capital charge per unit of working time. The annual interest *plus* depreciation on the plant is divided by the number of working hours per year, and further reduced to a capital charge per second by dividing by 3600. This is taken as an initial constant not varying with the size of the turbine *plus* a part $= aA$ proportional to this size as measured by the gate-area A. Adding to this the total working expenses, taken as in the other problems stated above, the total cost of the power per second is

$$K = C + aA + p\frac{H}{\epsilon},$$

the constant C being different from that in the previous equations. Here H, p, and a are also constants, but A and ϵ vary together according to the angle γ chosen for the guide-blades. The larger this angle the less is the efficiency, but the less also is the size of turbine required to develop the prescribed H. Let K′, A′, and ϵ' be the γ-gradients of K, A, and ϵ. The above equation gives

$$K' = aA' - pH\frac{\epsilon'}{\epsilon^2}.$$

K is a minimum when $K' = 0$, and this gives

$$\epsilon^2\frac{A'}{\epsilon'} = \frac{p}{a}H,$$

which, expressed in terms of γ, gives the criterion

$$\frac{1 - 3\sin^2\gamma}{1{\cdot}94\sin^3\gamma} = w^3\frac{p}{a}.$$

From this equation both H and A have disappeared by elimination, showing that the angle γ giving maximum commercial economy does not vary with the horse-power required, and is the same for large and small sizes of turbine. In finding it the water entrance-velocity w has been assumed constant, which practically means that the available head of water is fixed. The best γ depends on the cube of this velocity, and on the ratio of the extra working costs per extra unit of water-power consumed to the extra capital costs per extra square foot of gate-area in the size of turbine. The following are the numerical results of the formula:—

γ	10°	15°	20°	25°	30°	35°
$w^3\frac{p}{a}$	87	23·1	8·1	3·1	1·00	·034
$\frac{pH}{aA\epsilon}$	15·1	5·97	2·73	1·30	·50	·02
$\frac{aA}{pH}$	·068	·180	·415	·94	2·67	75·4

Appendix K.—Commercial Economy.

(End of Chap. IX., p. 117.)

In all industrial applications of the doctrine of maximum and minimum values, it is most important to remember that the *exact* attainment of the *exact* maximum or minimum is never of practical importance. The method of the calculus here explained applies only to quantities with *continuous* variation; and, as the gradient is zero at the place of maximum or minimum, on either side of this *exact* place there is a considerable range of base condition throughout which the deviation from maximum or minimum is so small as to be of no consequence. Indeed, in industrial problems the really best adjustments are hardly ever coincident with the exact values found theoretically, and for this reason that the theory never includes the consideration of every influential element. Minor elements are left out of the account in the theoretical calculation, and when the theoretical result has been found these minor elements quite rightly indicate the advisability of a small deviation from it in one or the opposite direction.

In the author's work, *Commercial Economy in Steam and other Heat Power-Plants,* published in 1905, many very interesting physical and financial maximum problems in the economic use of steam are worked out.

In this work the author enunciates for the first time a definite measure of industrial economy applicable to all productive effort. To this the name "economy coefficient" is given. It is $\frac{P}{CT}$, where P equals the value of any quantity of the product, C the cost of production of the same, and T the "time of turn over." If P be taken as the *time-rate* of production reckoned at its final value, then CT will be the "working capital" permanently held up in the maintenance of the manufacture; so that the economy coefficient may also be expressed as "Value of Annual Production ÷ Working Capital." This *working* capital does not include the fixed capital sunk in plant, buildings, etc.

The economy thus measured is capable of being raised or lowered by changes of various kinds in the methods of production. If such change affect all three factors P, C, and T, and if the change be capable of being made gradually, then the concurrent rates of change of these factors may be called P′, C′, and T′, these being, say, the x-gradients, if x be the measure of the element of manufacture which is being varied. Maximum economy is reached, *so far as it is affected by change of x,* when the x-gradient of $\frac{P}{CT}$ is zero; that is, when

$$\frac{P'}{P}=\frac{C'}{C}+\frac{T'}{T}.$$

This criterion of maximum commercial economy is quite general in its applicability to every kind of productive industry in its development by every sort of change capable of continuous gradation—whether in the manufacturing experiments the change be actually made gradually or suddenly.

If P be the quantity produced per unit time and ϵ a coefficient of physical efficiency giving the ratio of this product to the quantity of raw material consumed, so that $\frac{P}{\epsilon}$ is this latter quantity; if w be the total cost *per extra unit* of such raw material consumed together with the cost of working it up to the condition of the finished product; and if p be the final value *per extra unit* of the product P; then, if p and w are constant, any modification of the manufacture which affects P and ϵ concurrently, gives a rate of variation of the net revenue

$$R' = \frac{wP}{\epsilon}\left\{\frac{P'}{P}\left(\frac{p}{w}\epsilon - 1\right) + \frac{\epsilon'}{\epsilon}\right\}.$$

Thus the maximum revenue is obtained when the rate of production is adjusted so as to make

$$-\frac{\epsilon'/\epsilon}{P'/P} = \frac{p}{w}\epsilon - 1.$$

Here the size and character of plant is supposed fixed, and this equation gives the most commercially economic rate at which to work the given plant.

The other most important commercial problem is to determine the best size of plant for a prescribed rate of production P. Here P is constant ($P' = 0$). With a larger or more expensive plant the efficiency may be raised so as to lessen the working expenses in proportion to $\frac{wP}{\epsilon}$, but at the same time the capital charges are raised. These capital charges may be taken as equal to an initial constant *plus* $kPf(\epsilon)$ where k is a constant factor and $f(\epsilon)$ is a function of the efficiency dependent on the kind of industry investigated. The total annual cost for the prescribed rate of production P is thus

$$= \text{Constant} + kPf(\epsilon) + \frac{wP}{\epsilon},$$

and this is made a minimum by the adjustment

$$\epsilon^2 f'(\epsilon) = \frac{w}{k}.$$

w and k being among the prescribed data, and $f(\epsilon)$ and therefore ϵ being functions of the size or prime cost of the plant, this equation determines the most commercially economic size of plant to use. A particular example of the calculation has been given in Appendix I.

Appendix L.—Indeterminate Forms.

(End of Chap. IX., p. 117.)

Whatever meaning be attached to the symbol ∞, the ratio $\frac{0}{\infty}$ is clearly and definitely 0 or zero; while the ratio $\frac{\infty}{0}$ is definitely ∞. But the three quantities $\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty$ are more difficult to evaluate. They are termed "indeterminate." They arise as ratios and products of variables or fluxions, when these variables take special values, the said ratios and products having no ambiguity or indeterminateness when the variables have other than these special values. Thus X and $\mathfrak{X}$ may be functions of x, both of which come to zero for some special value of x.

There is in reality no such thing as a ratio between two zeros; a ratio can exist only between two quantities, and zero is not a quantity. The meaning attached to the symbol $\frac{0}{0}$ must, therefore, be in a sense conventional. The meaning attached to it is the ratio of X to $\mathfrak{X}$ when these functions have any *corresponding* or *simultaneous* minutely small values. To give this meaning real significance both X and $\mathfrak{X}$ must pass through zero as *continuous* functions of x; therefore they can both be represented graphically by curves on a scaled diagram. Let fig. 31 illustrate such graphic representations. Both X and $\mathfrak{X}$ curves cross the horizontal axis at the same point x_1. Draw tangents to the two curves at this point. These tangents coincide with the curves for minutely small distances on either side of the touching point, and all minutely small values of X and of $\mathfrak{X}$ are given equally well by the curves or by their tangents. The slopes of the tangents are X' and $\mathfrak{X}'$ taken at x_1, written, say, X'_1 and $\mathfrak{X}'_1$. For any small $\pm\delta x$ on either side of x_1, the values of X and $\mathfrak{X}$ are thus $X'_1\delta x$ and $\mathfrak{X}'_1\delta x$. Concurrent values of X and $\mathfrak{X}$ are those in which the δx is the same in both. It follows immediately that at any point close to x_1 on either side of it

$$\left(\frac{X}{\mathfrak{X}}\right)_1 = \frac{X'_1}{\mathfrak{X}'_1}.$$

As neither X'_1 nor $\mathfrak{X}'_1$ is zero or ambiguous in value, $\left(\frac{X}{\mathfrak{X}}\right)_1$, or $\frac{0}{0}$ according to the meaning above assigned to this symbol, can be

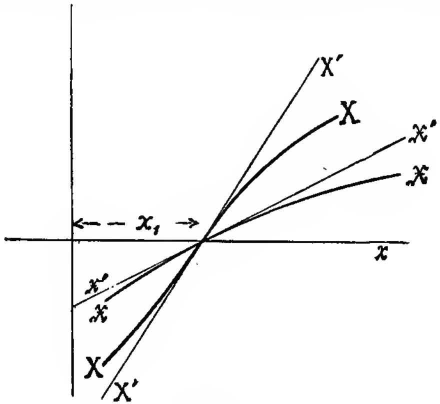

FIG. 31.

evaluated as the ratio of the two x-gradients at the particular value x_1.

In this demonstration it is assumed that both curves X and $\mathfrak{X}$

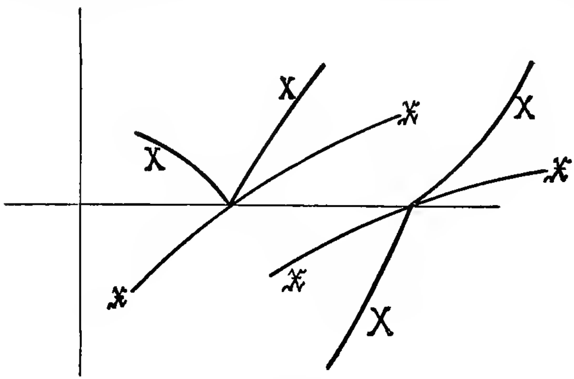

FIG. 32.

pass through the zero axis *without break of gradient*, that is, that both X′ and $\mathfrak{X}'$ are *continuous*. In fig. 32 are shown two pairs of curves in which, while the functions X and $\mathfrak{X}$ are themselves both continuous, the gradient of one of them, X′, is discontinuous at x_1. In this case $\frac{X'}{\mathfrak{X}'}$, and, therefore, also $\frac{X}{\mathfrak{X}}$, has a certain definite value for any $+\delta x$ beyond x_1 and another different definite value for any $-\delta x$ below x_1.

If $\frac{X'_1}{\mathfrak{X}'_1}$ also assumes the form $\frac{0}{0}$, similar reasoning shows that both it and $\left(\frac{X}{\mathfrak{X}}\right)_1$ have the same value as $\frac{X''}{\mathfrak{X}''}$ and can be evaluated by finding the ratio of these second gradients. The best graphic demonstration of this is obtained from a diagram with x as base and X′ and $\mathfrak{X}'$ as ordinates to two curves. Then X and $\mathfrak{X}$ for any $\pm\delta x$ on either side of x_1 are the small triangular areas under the curves with common base $\pm\delta x$. These areas are proportional to X′ and $\mathfrak{X}'$, and, therefore, also to X″ and $\mathfrak{X}''$.

If at x_1 the value of X becomes ∞ and that of $\mathfrak{X}$ zero, then construct an x-diagram with two curves giving $\frac{1}{X}$ and $\mathfrak{X}$. At x_1 the $\frac{1}{X}$ curve, as also the $\mathfrak{X}$ curve, both cross the zero axis, and the above rule may serve to evaluate $\left(\frac{\mathfrak{X}}{1/X}\right)_1 = (X\mathfrak{X})_1 = \infty \times 0$.

Since
$$\left(\frac{1}{X}\right)' = -\frac{X'}{X^2}, \text{ we have}$$
$$(X\mathfrak{X})_1 = -\left(X^2\frac{\mathfrak{X}'}{X'}\right)_1;$$

from which can also be deduced

$$(X\mathfrak{X})_1 = -\left(\mathfrak{X}^2\frac{X'}{\mathfrak{X}'}\right)_1.$$

But neither of these last two formulas is useful, because if X becomes ∞ at any finite value of x, so also does X′. The function $\left(\frac{1}{X}\right)$, however, may often be differentiated in terms of x so as to eliminate entirely both X and X′. Thus

$$(X\mathfrak{X})_1 = \infty \times 0 = \frac{\mathfrak{X}'_1}{\left(\frac{1}{X}\right)'_1}$$

usually gives a definite value. Otherwise the product $X\mathfrak{X}$ may reduce by cancellation to a function of x which does not give an indeterminate value at x. This latter method must be adopted when $X = x$, because $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$ gives $(x\mathfrak{X})_\infty = -x^2\mathfrak{X}' = -\infty^2 \times 0$, since, if $\mathfrak{X} = 0$ at $x = \infty$, necessarily $\mathfrak{X}'$ also $= 0$ at same limit.

To engineers the most interesting case is that of the commonly used expansion curve $pv^n = \text{K}$. This gives infinite volume for zero pressure. Here $v = \left(\frac{\text{K}}{p}\right)^{\frac{1}{n}}$ and $pv = \text{K}v^{1-n} = \text{K}^{\frac{1}{n}}p^{\frac{n-1}{n}}$.

In this case at zero pressure,

$$\begin{aligned} pv &= 0 \quad \text{if } n > 1 \\ &= k \quad \text{,, } n = 1 \\ &= \infty \quad \text{,, } n < 1, \end{aligned}$$

the last case being for a curve lying *above* the hyperbolic or "gas isothermal." This last curve corresponds to expansion accompanied by very rapid heating. The work done by the expansion down to zero pressure from p_1v_1 is (see § 110, p. 63)

$$\begin{aligned} \text{W} = \frac{p_1v_1 - (pv)_{p=0}}{n-1} &= \frac{p_1v_1}{n-1} \text{ if } n > 1 \\ &= \infty \quad \text{,, } n < 1 \end{aligned}$$

When $n = 1$, W, according to this formula, takes the indeterminate form $\frac{0}{0}$; but the special integration

$$\int_1^2 pdv = p_1v_1 \log \frac{v_2}{v_1}$$

shows that the value is ∞ when $p_2 = 0$, and $\therefore v_2 = \infty$.

If $\text{X}_1 = \infty$ and $\mathfrak{X}_1 = \infty$, then $\frac{\infty}{\infty} = \frac{\text{X}_1}{\mathfrak{X}_1} = \frac{1}{\mathfrak{X}_1} \Big/ \frac{1}{\text{X}_1} = \frac{0}{0}$;

to which the first rule given applies. Taking the ratio of the x-gradients, there is found

$$\left(\frac{\text{X}}{\mathfrak{X}}\right)_1 = \left(\frac{\text{X}}{\mathfrak{X}}\right)_1^2 \cdot \frac{\mathfrak{X}'_1}{\text{X}'_1}, \text{ or } \left(\frac{\text{X}}{\mathfrak{X}}\right)_1 = \frac{\text{X}'_1}{\mathfrak{X}'_1}.$$

As it stands this is of no use, because both X'_1 and $\mathfrak{X}'_1 = \infty$. But by cancellation, or otherwise, the ratio $\frac{X'_1}{\mathfrak{X}'_1}$ may be reducible to determinate form, while $\frac{X}{\mathfrak{X}}$ is not so.

A theoretically important case is the value of $x \log_e x$ when $x=0$. It takes then the form $-0 \times \infty$. Taking $x \log_e x = \frac{\log_e x}{1/x}$, and differentiating both numerator and divisor, we find the value $-\frac{1}{x \times \frac{1}{x^2}} = -x = 0$ at the limit.

The discussion in § 104, page 60, affords another illustration of the theoretical importance of this method. Here $\frac{x^n - 1}{n}$ was demonstrated graphically to equal $\log_e x$ when $n=0$, at which value it takes the form $\frac{0}{0}$. Considering x constant, the n-gradient of $(x^n - 1)$ is $x^n \log_e x$, and the n-gradient of n is 1. Since $x^0 = 1$, the method now explained gives at once $\frac{x^n - 1}{n} = \log_e x$ when $n=0$.

PART II.

CLASSIFIED REFERENCE TABLES

OF

INTEGRALS AND METHODS OF INTEGRATION,

IN ELEVEN SECTIONS.

PART II.

CLASSIFIED REFERENCE TABLES

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INTEGRALS AND METHODS OF INTEGRATION,

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APPENDICES.

TABLE OF CONTENTS.

NOTATION.

Letters near the beginning of the alphabet denote constants which may in general be positive or negative, whole or fractional, real quantities or numbers. Those near the end of the alphabet denote variables.

The symbol $\equiv$ stands for "**denotes**" or "**is identical with.**"

The symbols $>$, $\ngtr$, $<$, $\nless$ stand for **greater than, not greater than, less than, not less than**, respectively.

The symbols $f(\ )$, $F(\ )$, $\phi(\ )$, $\psi(\ )$, denote **any** function of the quantity placed inside the brackets, except when restricted by the context.

X, Ξ, $\mathfrak{X}$, χ are briefer symbols for functions of x.

L, M, N, P, Q, R also sometimes denote functions of the variable.

$Y \equiv$ any function of the variable y which is independent of x.

$f(a) \equiv$ the same function of a that $f(x)$ is of the variable x.

$$\Big[f(x)\Big]_a^b \equiv f(b) - f(a)$$

$$f'(x) \equiv \frac{df(x)}{dx} : \quad f''(x) \equiv \frac{d^2f(x)}{dx^2} : \text{----} f^n(x) \equiv \frac{d^nf(x)}{dx^n}$$

$$X' \equiv \frac{dX}{dx} : \quad X'' \equiv \frac{d^2X}{dx^2} : \text{----} X^{(n)} \equiv \frac{d^nX}{dx^n}$$

$$Y' \equiv \frac{dY}{dy} : \quad Y'' \equiv \frac{d^2Y}{dy^2} : \text{----} Y^{(n)} \equiv \frac{d^nY}{dy^n}$$

$$f'(X) \equiv \frac{df(X)}{dX} : \quad f'(Y) \equiv \frac{df(Y)}{dY}$$

$$f^{\prime}_{x}(X) \equiv \frac{df(X)}{dx} : \quad f^{\prime\prime}_{x}(X) \equiv \frac{d^2f(X)}{dx^2} : \text{----} f^{n}_{x}(X) \equiv \frac{d^nf(X)}{dx^n}$$

$\dfrac{\partial}{\partial x} \equiv$ partial differentiation with respect to x.

$f^{\prime}_{x}(x,y) \equiv$ the differential coefficient with respect to x of the function $f(x,y)$ where x and y are independent variables.

$f^{\prime\prime}_{xy}(x,y) \equiv f^{\prime\prime}_{yx}(x,y) \equiv$ the second differential coefficient of $f(x,y)$ with respect to x and y, where x and y are independent variables.

$\int X dx \equiv$ Integral of X with respect to x

$$\int^{(2)} X dx^2 \equiv \int\int X dx^2 \equiv \int\int X dx dx \equiv \int \left\{ \int X dx \right\} dx$$

$$\int^{(n)} X dx^n \equiv \int\int \text{--} (n \int_{\text{symbols}}) X dx^n \equiv \int\int \text{--} (n \int_{\text{symbols}}) X dx dx \text{--} (n dx\text{'s})$$

$$\int\int f(x,y) dy dx \equiv \int \left\{ \int f(x,y) dy \right\} dx$$

Σ or $\Sigma(\;) \equiv$ Sum of a series of terms of the same type as that following Σ or placed within the brackets.

$\int_a^b f(x)dx \equiv \left[\int f(x)dx \right]_a^b \equiv$ the limiting value towards which $\left[\Sigma f(x)\delta x \right]_a^b$ approaches as δx approaches the limiting value 0.

$$\int_a f(x)dx \equiv \int_a^x f(x)dx$$

$n! \equiv 1 \times 2 \times 3 \text{----} (n-1) \times n$; ($n$ being a positive integer).

$\log_{10} x \equiv \log x$ in the Decimal or Common system of logarithms.

$\log_e x \equiv \log x$ in the Natural or Neperian „ „ .

$e \equiv 2{\cdot}7182818 \text{----} \equiv 1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \text{----} \equiv$ base of Neperian logarithms

$$exp(x) \equiv e^x \equiv 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \text{----}$$

B_1, B_2 ---- (Bernoulli's Numbers).*

In trigonometric functions the angles are given in "circular measure," the unit being the radian which $= \frac{180°}{\pi}$ or 57° 17′ 45″ nearly.

$\Gamma(n) \equiv \int_0^\infty e^{-x} x^{n-1} dx$ (the Gamma Function). See X. 1–6.

ABBREVIATIONS.

Sp. Case $\equiv$ Special Case

Exc. $\equiv$ Exceptional Case, or Case of failure.

* See *note* at end of Section VIII.

I. GENERAL THEOREMS.

1. $$\int X' dx = X + C$$

$$\int^{(2)} X'' dx^2 = X + C_1 x + C_0$$

$$\int^{(3)} X''' dx^3 = X + C_2 x^2 + C_1 x + C_0$$

$$\int^{(n)} X^{(n)} dx^n = X + C_{n-1} x^{n-1} + C_{n-2} x^{n-2} \cdots + C_1 x + C_0$$

2. $$\int_a^b f'(x) dx = f(b) - f(a)$$

3. $$\int a f(x) dx = a \int f(x) dx$$

4. $$\int (X \pm \mathfrak{Y} \pm \mathfrak{Z} \cdots) dx = \int X dx \pm \int \mathfrak{Y} dx \pm \int \mathfrak{Z} dx \cdots$$

5. $$\int_a^c X dx = \int_a^b X dx + \int_b^c X dx$$

Sp. Case: $$\int_a^b X dx = -\int_b^a X dx$$

6. $$\int_{x_1}^{x_2} f(x) dx = \int_{X_1}^{X_2} f\{\phi(X)\} \phi'(X) dX$$

where $x \equiv \phi(X)$, $x_1 \equiv \phi(X_1)$, $x_2 \equiv \phi(X_2)$ $\qquad \phi'(X) \equiv \dfrac{d\phi(X)}{dX} \equiv \dfrac{dx}{dX}$

Sp. Case: $$\int X dx = \int \frac{X}{X'} dX$$

7. (INTEGRATION BY PARTS.)

$$\int X\Xi dx = X\int \Xi dx - \int \left\{ X' \int \Xi dx \right\} dx$$

$$\text{or} \int X\Xi' dx = X\Xi - \int X'\Xi dx = X\Xi - \int \Xi dX$$

$$\text{Sp. Case}: \int X dx = Xx - \int x dX.$$

APPENDIX M.—INTEGRATION BY PARTS: SPECIAL CASES.

(I. No. 7.)

The special case $\Xi' = x^m$ is worth notice. It gives

$$\int X x^m dx = \frac{X x^{m+1}}{m+1} - \frac{1}{m+1}\int X' x^{m+1} dx.$$

With $$m = 1 : \int X x dx = \frac{X x^2}{2} - \frac{1}{2}\int X' x^2 dx.$$

" $$m = 0 : \int X dx = Xx - \int X' x dx.$$

" $$m = -1 : \int \frac{X}{x} dx = X \log x - \int X' \log x dx.$$

With

$$X = (\log x)^n : \int x^m (\log x)^n dx = \frac{x^{m+1} (\log x)^n}{m+1} - \frac{n}{m+1}\int x^m (\log x)^{n-1} dx.$$

With $$X = \log x : \int x^m \log x dx = \frac{x^{m+1}}{m+1}\left(\log x - \frac{1}{m+1}\right).$$

" $$X = \log x : \int \Xi' \log x dx = \Xi \log x - \int \frac{\Xi}{x} dx.$$

All the formulæ of Section IX. are deduced by help of this I. 7.

8. $\int X\Xi dx = X\int \Xi dx - X'\int^{(2)}\Xi dx^2 + X''\int^{(3)}\Xi dx^3 + \cdots$

$$\pm X^{(n-1)}\int^{(n)}\Xi dx^n \mp \int \left\{ X^{(n)}\int^{(n)}\Xi dx^n \right\} dx$$

Sp. Case: $\Xi = 1 : \int X dx = \frac{x}{1}X - \frac{x^2}{1\cdot 2}X' + \frac{x^3}{3!}X'' - \frac{x^4}{4!}X''' + \text{etc.}$

9. $\int \frac{X'}{X} dx = 2\cdot3026 \cdots \log_{10} X + C = \log_e X + C$

10. APPROXIMATE INTEGRATION.

(i.) $\int_a^b X dx = \frac{b-a}{2n}(X_0 + 2X_1 + 2X_2 + \cdots + 2X_{n-1} + X_n)$

where X_0 is the value of X when $x = a$

„ X_1 „ „ „ „ $= a + \frac{b-a}{n}$

„ X_2 „ „ „ „ $= a + \frac{2(b-a)}{n}$

„ X_r „ „ „ „ $= a + r.\frac{b-a}{n}$

with $(b-a)$ divided into n equal parts giving $(n+1)$ values of X.

(ii.) (Simpson's Rule.)

$$\int_a^b X dx = \frac{b-a}{6n}\left\{ X_0 + X_{2n} + 4(X_1 + X_3 + X_5 + \cdots) + 2(X_2 + X_4 + X_6 + \cdots + X_{2n-2}) \right\}$$

where X_0, X_1, etc. $\equiv$ as in (i.);

with $(b-a)$ divided into $2n$ equal parts giving $(2n+1)$ values of X.

(iii.) $$\int_a^b f(x)dx = h\left[\tfrac{1}{2}\{f(b) + f(a)\} + f(a+h) + f(a+2h) + \cdots + f(b-h)\right]$$

$$+ \left[-\frac{B_1 h^2}{2!}f'(x) + \frac{B_2 h^4}{4!}f'''(x) - \frac{B_3 h^6}{6!}f^{(5)}(x) + \cdots \right]_a^b .*$$

* See *note* at end of Sect. VIII.

11. Method of Undetermined Coefficients.

(i.) If a function be expressible in a certain form containing unknown coefficients, these coefficients can be determined by transforming the identity and equating coefficients of terms whose variable part is the same function of the variable (*e.g.*, the same power, or the same trigonometrical function), or in which the variable is absent (constant terms).

The transformation referred to may be:—

(*a*) Differentiating both sides (as in III. B. 18).

(*b*) Clearing of fractions (as in II. E. 2, etc.).

(ii.) Another method: Give to the primitive variable as many different values, in the identity, as there are coefficients to be determined; whereby we get as many equations as are necessary to determine them.

General Note *re* Imaginaries.

Some of the formulas here given contain parts which would become imaginary if the quantities involved took values outside certain limits: becoming, *e.g.*, square roots and logarithms of negative quantities, inverse sines of quantities greater than unity, etc. When a *definite* integral is deduced from the indefinite one, these imaginaries, explicitly or implicitly, cancel one another, if the subject of integration is itself real. But, in many instances, two or more forms are given for the integral of the same function (*e.g.*, III. B. 6, VI. 5, 6), of which that one is to be selected which, for the values of the constants in the particular problem under consideration, is free from imaginaries.

Some of these formulas contain parts which are imaginary for certain values of x only, whatever the constants may be, and others do so for all values of x when the constants are outside certain limits. *E.g.*, the formula $\sin^{-1}\frac{x}{a}$ is imaginary when $x>a$, but not when x lies between $-a$ and $+a$. On the other hand, the formula $\frac{1}{\sqrt{b}}\sinh^{-1}\left\{x\sqrt{\frac{b}{a}}\right\}$ contains imaginaries when b is negative, whatever the value of x may be.

In these classified tables, the conditions under which a formula involves imaginaries are, as a rule, pointed out in cases of the latter sort, but not in those of the former.

Special Note as to Logarithmic Terms.

When a term of the form A log X, where A is a constant, occurs in an integral, it becomes imaginary when x has such values as make X negative; but in such cases A log $(-X)$, which is real, may always be used instead of A log X, since it has the same differential coefficient as A log X. This note applies to III. A. 3, 5, 11, 16, etc.

Note *re* Inverse Use of Tables to find Differential Coefficients.

Although the chief purpose of these Tables is to assist in Integration, they may also be used to find gradients or differential coefficients of given functions. To use them for this purpose, search for the given function on the right-hand side of the page. Its x-gradient is the corresponding quantity on the left-hand side of the page with the sign of integration $\int$ and dx removed.

The function to be differentiated will not, however, always be found under the subject-title proper to the function, since the arrangement of the tables classifies differentials, and not integrals, according to subject. For instance, the differentials of $\sin^{-1}x$, $\tan^{-1}$, etc., $\sinh^{-1}x$, $\cosh^{-1}x$, etc., will be found in Section III.—"Algebraical."

II.—CHIEF METHODS OF TRANSFORMATION.

A. Express the subject of integration as the sum of a series of terms, and integrate these separately (see I. 4). (Integration by decomposition or separation.)

$$E.g., \int \log\{(1+2x)(1+3x)\}dx \equiv \int \{\log(1+2x)+\log(1+3x)\}dx$$
$$\equiv \int \log(1+2x)dx + \int \log(1+3x)dx.$$

B. Add and subtract the same quantity. *E.g.*,

$$\int \frac{xdx}{1+2x} = \int \frac{(x+\frac{1}{2})-\frac{1}{2}}{1+2x}dx$$
$$= \int \left\{ \frac{1}{2} - \frac{1}{2(1+2x)} \right\} dx.$$

C. Multiply (or divide) numerator and denominator by the same quantity. *E.g.*,

$$\int \tan^3 x dx = \int \frac{\tan^3 x(1+\tan^2 x)dx}{1+\tan^2 x}$$
$$= \int \frac{\tan^3 x d \tan x}{1+\tan^2 x}.$$

Sp. Case:

$$\frac{M+N\sqrt{R}}{m+n\sqrt{R}} = \frac{(M+N\sqrt{R})(m-n\sqrt{R})}{m^2-n^2R}$$

D. Expand in a series (see p. 146). *E.g.*,

$$\int \frac{dx}{\sqrt{(1-k^2\sin^2 x)}} = \int \left\{ 1 + \frac{1}{2}k^2\sin^2 x + \frac{1\cdot 3}{2\cdot 4}k^4\sin^4 x + \cdots \right\} dx.$$

Sp. Case: II. F.

E. Resolve rational fractions into partial fractions. (See p. 167.)

F. Express a product of powers of sines and cosines as a sum of terms, each consisting of a sine or cosine multiplied by a constant. (See p. 168.)

G. Substitute $f(\mathrm{X})$ for x and $f'(\mathrm{X})d\mathrm{X}$ for dx. (See I. 6.)

$$\text{Sp. Case : } \left.\begin{array}{r} f(\mathrm{X}) \equiv p\mathrm{X} + q = x \\ dx = pd\mathrm{X} \end{array}\right\}.$$

In the case of a definite integral, change the limits correspondingly (See I. 7), or else transform back to x after integration, before assigning limits. (See p. 148.)

H. Differentiate or integrate an integral with respect to any quantity in it which is not a function of x, and a new integral is deduced.

K. Use integration by parts. (See I. 7.)

II. D. *Chief Methods of Expansion in Series* :—

1. Binomial Theorem.
2. Exponential Theorem.
3. Expansion of $\text{Log}_e(1 \pm x)$.
4. Trigonometric series derived from the preceding expansions, by use of imaginaries.
5. Taylor's Theorem or Maclaurin's Theorem (including 1, 2, 3, as special cases).
6. Fourier's Expansion in series of sines and cosines.
7. Spherical Harmonics, Lamé's Functions, Bessel's Functions, Toroidal Functions, etc.

II. E. *Partial Fractions.*

$$\frac{F(x)}{f(x)} \equiv \frac{Ax^m + Bx^{m-1} + \cdots + H}{ax^n + bx^{n-1} + \cdots + h}$$ where m and n are positive integers.

1. If $m \nless n$ reduce $\frac{F(x)}{f(x)}$ by ordinary division to an integral function of x, + a fraction of similar form to the above with $m < n$.

2. *First Case.* $f(x) \equiv$ a product of simple factors, all different.

$$f(x) \equiv a(x-p)(x-q) \cdots (x-s)$$
$$\frac{F(x)}{f(x)} \equiv \frac{P}{x-p} + \frac{Q}{x-q} + \cdots + \frac{S}{x-s}.$$

To find P, Q ---- S, which are constants, use I. 11 (i.), (*b*).

Otherwise, $P = \frac{F(p)}{f'(p)}$, $Q = \frac{F(q)}{f'(q)}$ etc.

3. *Second Case.* Some factors repeated.

E.g. $f(x) \equiv a(x-p)^3(x-q)^2(x-r) \cdots$

$$\frac{F(x)}{f(x)} \equiv \frac{P_1}{x-p} + \frac{P_2}{(x-p)^2} + \frac{P_3}{(x-p)^3} + \frac{Q_1}{x-q} + \frac{Q_2}{(x-q)^2} + \frac{R}{x-r} + \cdots$$

To find P_1 P_2 ---- use I. 11. (i.).

4. *Third Case.* $f(x)$ not a product of real simple factors, but contains quadratic factors all different. *E.g.*

$$f(x) \equiv a(x^2 + kx + l)(x^2 + mx + n)(x-p)(x-q) \cdots$$
$$\frac{F(x)}{f(x)} \equiv \frac{Kx + L}{x^2 + kx + l} + \frac{Mx + N}{x^2 + mx + n} + \frac{P}{x-p} + \cdots$$

Determine the constants K, L, M, etc., by I. 11. (i.).

5. *Fourth Case.* Some quadratic factors repeated. *E.g.*

$$f(x) \equiv a(x^2 + kx + l)^3(x^2 + mx + n)(x-p) \cdots$$
$$\frac{F(x)}{f(x)} \equiv \frac{K_1x + L_1}{x^2 + kx + l} + \frac{K_2x + L_2}{(x^2 + kx + l)^2} + \frac{K_3x + L_3}{(x^2 + kx + l)^3}$$
$$+ \frac{Mx + N}{x^2 + mx + n} + \frac{P}{x-p} + \cdots$$

Determine the constants K_1 L_1 etc., by I. 11. (i.).

II. F. *Sines and Cosines.*

1. Use repeatedly

$$\text{Sin}\, mx \cos nx \equiv \frac{1}{2}\left\{\sin(m+n)x + \sin(m-n)x\right\}$$

$$\text{Cos}\, mx \cos nx \equiv \frac{1}{2}\left\{\cos(m+n)x + \cos(m-n)x\right\}$$

$$\text{Sin}\, mx \sin nx \equiv \frac{1}{2}\left\{\cos(m-n)x - \cos(m+n)x\right\}$$

E.g. $\text{Sin}^3 x \cos 2x \cos x \equiv \frac{1}{8}\sin 4x - \frac{1}{16}\sin 6x - \frac{1}{16}\sin 2x.$

2. Alternative method. $i \equiv \sqrt{-1}.$

$$\text{Use X} \equiv e^{ix} \therefore 2\cos x = X + X^{-1}$$
$$2i \sin x = X - X^{-1}$$
$$2\cos nx = X^n + X^{-n}$$
$$2i \sin nx = X^n - X^{-n}.$$

Express sines and cosines of x or its multiples in terms of X. Multiply out. Collect pairs of terms of the form C $(X^n \pm X^{-n})$, and reintroduce sines and cosines.

E.g. $\text{Sin}^3 x \cos 2x \cos x$

$$= \frac{1}{(2i)^3 2^2}(X - X^{-1})^3 (X^2 + X^{-2})(X + X^{-1})$$

$$= -\frac{1}{16}\frac{1}{2i}\{X^6 - X^{-6} - 2(X^4 - X^{-4}) + X^2 - X^{-2}\}$$

$$= -\frac{1}{16}\{\sin 6x - 2\sin 4x + \sin 2x\}.$$

II. G.—*Substitutions.*

1. $(ax+b)^n dx = \frac{1}{a} X^n dX$

2. $F(ax+b)dx = \frac{1}{a} F(X)dX$

$\left.\right\} X \equiv ax+b.$

Sp Case: $b = \pm \frac{\pi}{2}$. (See VI. head note.)

3. $\frac{dx}{x\sqrt{(ax^2+b)}} = \frac{-dX}{\sqrt{(a+bX^2)}}$ $X \equiv \frac{1}{x}.$

4. $\frac{x^m dx}{(ax+b)^{q/p}} = \frac{p}{a^{m+1}}(X^p - b)^m X^{p-q-1} dX$

$X \equiv (ax+b)^{1/p}.$

5. $F(ax^2+bx+c)dx = F\{a(X^2+k)\}dX$

$X \equiv x + \frac{b}{2a}, \quad k \equiv \frac{4ac-b^2}{4a^2}.$

6 $\frac{dx}{\sqrt{\{(x-a)(x-b)\}}} = \frac{2dX}{1-X^2}, \; X \equiv \sqrt{\left(\frac{x-a}{x-b}\right)}.$

7. $F(x^2+k^2)dx = F(k^2\sec^2 X)k\sec^2 X dX$

$X \equiv \tan^{-1}\frac{x}{k}.$

Otherwise $= F(k^2\cosh^2 X)k\cosh X dX$

$X \equiv \sinh^{-1}\frac{x}{k}.$

8. $F\{(x^2-k^2)^{\frac{1}{2}}\}dx = F(k\tan X)k\sec X\tan X dX$

$X \equiv \sec^{-1}\frac{x}{k}.$

Otherwise $= F(k\sinh X)k\sinh X\,.\,dX$

$X \equiv \cosh^{-1}\frac{x}{k}.$

9. $F\{(k^2-x^2)^{\frac{1}{2}}\}dx = F(k\cos X)k\cos X dX$

$X \equiv \sin^{-1}\frac{x}{k}.$

10. $F(x, \log_e x)dx = F(e^X, X)e^X dX$

$$X \equiv \log_e x.$$

11. $F(b\cos x + c\sin x)dx = F\{\sqrt{(b^2+c^2)}\sin X\}dX$

$$X \equiv x + \tan^{-1}\frac{b}{c}.$$

Otherwise, as in II. G. 12.

12. $F(a + b\cos x + c\sin x)dx = F\left\{\frac{a+b+2cX+(a-b)X^2}{1+X^2}\right\}\frac{2dX}{1+X^2}$

$$X \equiv \tan\frac{x}{2}.$$

13. $F(a + b\cos^2 x + c\sin^2 x)dx = F\left\{\frac{a+b+(a+c)X^2}{1+X^2}\right\}\frac{dX}{1+X^2}$

$$X \equiv \tan x.$$

14. $F(\cos x, \sin^2 x).\sin x dx = -F\{X, (1-X^2)\}dX$

$$X \equiv \cos x.$$

15. $F(\sin x, \cos^2 x).\cos x dx = F\{X, (1-X^2)\}dX$

$$X \equiv \sin x.$$

16. $F(\sin^{-1} x)dx = F(X)\cos X dX,$

$$X \equiv \sin^{-1} x$$

and similarly for other inverse functions.

III.-IX.—TABLE OF INTEGRALS.

III. ALGEBRAIC FUNCTIONS.

III. A. *Mainly Rational.*

1. $\int a\,dx = C + ax.$

2. $\int x^n dx = C + \frac{x^{n+1}}{n+1}.$

[Exc. $n = -1$. (See III. A. 3.)]

3. $\int \frac{dx}{x} = C + 2{\cdot}302585 \text{----} \times \log_{10} x$

$= C + \log_e x.$

4. $\int (ax+b)^n dx = C + \frac{1}{(n+1)a}(ax+b)^{n+1}.$

Exc. $n = -1$. (See III. A. 5.)

5. $\int \frac{dx}{ax+b} = C + \frac{2{\cdot}3026\text{----}}{a} \times \log_{10}(ax+b)$

$= C + \frac{1}{a}\log_e(ax+b).$

6. $\int \frac{Ax+B}{ax+b}dx = C + \frac{A}{a}x + \frac{aB - Ab}{a^2}\log_e(ax+b).$

7. $\int x^m(ax+b)^n dx.$ Use II. A., or IX. A. 1, or III. A. 20.

8. $\int \frac{dx}{1+x^2} = C + \tan^{-1}x = C - \cot^{-1}x.$

9. $\int \frac{dx}{1-x^2} = C + \frac{1}{2}\log_e\frac{1+x}{1-x} = C + \tanh^{-1}x \quad [x<1]$

$= C + \frac{1}{2}\log_e\frac{x+1}{x-1} = C + \coth^{-1}x \quad [x>1].$

10. $\int \frac{dx}{ax^2+b} = \frac{1}{\sqrt{(ab)}}\tan^{-1}x\sqrt{\frac{a}{b}} + C$ when $ab>0$.

$= \frac{1}{\sqrt{(-ab)}}\int\frac{dX}{1-X^2}$ where $X \equiv x\sqrt{\left(-\frac{a}{b}\right)}$ when $ab<0$.

(See III. A. 9.)

11. $\displaystyle\int\frac{dx}{(Ax+B)(ax+b)}=C+\frac{1}{Ab-aB}\log_e\left(\frac{Ax+B}{ax+b}\right).$

12. $\displaystyle\int\frac{x^m dx}{1+x^n}$ where m is a positive integer or 0, and n a positive integer. If $m \nless n$, use II. E. 1; if $m<n$, thus:—

$$=C+\frac{(-1)^m}{n}\log_e(1+x)-\frac{1}{n}\Sigma\left\{\cos\frac{r(m+1)\pi}{n}\log_e\left(x^2-2x\cos\frac{r\pi}{n}+1\right)\right\}$$
$$+\frac{2}{n}\Sigma\left\{\sin\frac{r(m+1)\pi}{n}\tan^{-1}\left(\frac{x-\cos\frac{r\pi}{n}}{\sin\frac{r\pi}{n}}\right)\right\}.$$

N.B.—If n is odd, r takes the values 1, 3, 5 ---- $(n-2)$.
If n is even, r „ „ 1, 3, 5 ---- $(n-1)$, and the term $\frac{(-1)^m}{n}\log_e(1+x)$ is *omitted.*

13. $\displaystyle\int\frac{x^m dx}{1-x^n}$ where m is a positive integer or 0, and n is an odd positive integer.

$$=(-1)^{m+1}\int\frac{X^m dX}{1+X^n},\ \text{where } X\equiv -x.\quad \text{(See III. A. 12.)}$$

14. $\displaystyle\int\frac{x^m dx}{1-x^n}$ where m is a positive integer or 0, and n is an even positive integer.

$$=C+\frac{1}{n}\log_e(1-x)-\frac{(-1)^m}{n}\log_e(1+x)$$
$$-\frac{1}{n}\Sigma\left\{\cos\frac{r(m+1)\pi}{n}\log_e\left(x^2-2x\cos\frac{r\pi}{n}+1\right)\right\}$$
$$+\frac{2}{n}\Sigma\left\{\sin\frac{r(m+1)\pi}{n}\tan^{-1}\left(\frac{x-\cos\frac{r\pi}{n}}{\sin\frac{r\pi}{n}}\right)\right\}$$

where r takes the values 2, 4, 6 ---- $(n-2)$.

15. $\displaystyle\int\frac{x^m dx}{ax^n+b}=\frac{1}{b}\left(\frac{b}{a}\right)^{(m+1)/n}\int\frac{X^m dX}{1+X^n}$ where $X\equiv x\left(\frac{a}{b}\right)^{1/n}$, if $ab<0$.

See III. A. 12, if m and n are positive integers.

$$=\frac{1}{b}\left(-\frac{b}{a}\right)^{(m+1)/n}\int\frac{X^m dX}{1-X^n}\ \text{where } X\equiv x\left(-\frac{a}{b}\right)^{1/n},\ \text{if } ab<0.$$

See III. A. 13, 14, if m and n are positive integers. Otherwise, see IX. A. 1.

16. $$\int\frac{dx}{ax^2+bx+c}=C+\frac{2}{\sqrt{(-A)}}\tan^{-1}\frac{2ax+b}{\sqrt{(-A)}}\quad\text{where } A\equiv b^2-4ac, \text{ if } A<0$$

$$=C+\frac{1}{\sqrt{A}}\log_e\frac{2ax+b-\sqrt{A}}{2ax+b+\sqrt{A}}\qquad\text{if } A>0.$$

17. $$\int\frac{(Ax+B)dx}{ax^2+bx+c}=\frac{A}{2a}\log_e(ax^2+bx+c)+\frac{2aB-Ab}{2a}\int\frac{dx}{ax^2+bx+c}.$$

(See III. A. 16.)

18. $$\int\frac{Ax+B}{X^n}dx \text{ where } X\equiv ax^2+bx+c$$

$$=\frac{-A}{2(n-1)aX^{n-1}}+\frac{2Ba-Ab}{2a}\int\frac{dx}{X^n}.\qquad\text{(See IX. A. 4.)}$$

Sp. Case: $A=0$. (See IX. A. 4.)

19. $$\int\frac{Ax^m+Bx^{m-1}+\cdots+K}{ax^n+bx^{n-1}+\cdots+p}dx,$$ where m, n are positive integers

Reduce by II. E. to terms like these:—

$$\int Ax^q dx,\ \int\frac{Adx}{x-p},\ \int\frac{Adx}{(x-p)^r},\ \int\frac{Lx+M}{x^2+lx+m}dx,\ \int\frac{Lx+M}{(x^2+lx+m)^r}dx;$$

for which, see III. A. 2, 5, 4, 17, 18.

20. $$\int x^m(ax^n+b)^{p/q}dx=\frac{q}{na}\int X^{p+q-1}\left(\frac{X^q-b}{a}\right)^{(m+1)/n-1}dX$$

where $X\equiv(ax^n+b)^{1/q}$

$$=-\frac{q}{n}\int b^{(m+1)/n+p/q}(X^q-a)^{-(m+1)/n-p/q-1}X^{p+q-1}dX$$

where $X\equiv(bx^{-n}+a)^{1/q}$.

Use the former when $\frac{m+1}{n}$ is a positive integer.

" latter " $\frac{m+1}{n}+\frac{p}{q}$ is a negative integer.

In either case, expand the binomial factor and use II. A. and III. A. 2.

Definite Integrals with Numerical (or Particular) Limits.

21. $\int_0^\infty \frac{x^m dx}{1+x^n} = \frac{\pi}{n \sin \frac{(m+1)\pi}{n}}$ where n and m are even positive integers, and $m<n$.

22. $\int_0^1 \frac{dx}{\sqrt[n]{(1-x^n)}} = \frac{\pi}{n} \operatorname{cosec} \frac{\pi}{n}$ if $n>1$.

23. $\int_0^1 \frac{dx}{\sqrt[n]{(1+x^n)}} = \frac{\pi}{n} \cot \frac{\pi}{n}$ " "

24. $\int_0^1 \frac{(x^m+x^n)dx}{(1+x)^{m+n+2}} = \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}$. (See X. 1–6.)

25. $\int_0^1 x^m(1-x)^n dx = \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}$. " "

26. $\int_0^\infty \frac{x^m dx}{(1+x)^{m+n+2}} =$ " " "

27. $\int_0^a x^m(a-x)^n dx = a^{m+n+1}\frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}$ " "

III. B. *Quadratic Surds.*

1. $\int (ax+b)^{\frac{1}{2}} dx = C + \frac{2}{3a}(ax+b)^{\frac{3}{2}}.$

2. $\int f\{(ax+b)^{\frac{1}{2}}\} dx = \frac{2}{a}\int X f(X) d(X)$

where $X \equiv (ax+b)^{\frac{1}{2}}.$

Sp. Cases :—

(*1*) $\int x(ax+b)^{\frac{1}{2}} dx = C + \frac{2}{5a^2}(ax+b)^{\frac{5}{2}} - \frac{2b}{3a^2}(ax+b)^{\frac{3}{2}}.$

(*2*) $\int \frac{dx}{x(ax+b)^{\frac{1}{2}}} = 2\int \frac{dX}{X^2 - b}.$ (See III. A. 10.)

3. $\int \frac{dx}{(x^2+1)^{\frac{1}{2}}} = C + \log_e\{x + (x^2+1)^{\frac{1}{2}}\} = C + \sinh^{-1}x.$

4. $\int \frac{dx}{(x^2-1)^{\frac{1}{2}}} = C + \log_e\{x + (x^2-1)^{\frac{1}{2}}\} = C + \cosh^{-1}x.$

5. $\int \frac{dx}{(1-x^2)^{\frac{1}{2}}} = C + \sin^{-1}x = C - \cos^{-1}x.$

6. $\int \frac{dx}{(ax^2+b)^{\frac{1}{2}}} = C + \frac{1}{\sqrt{a}}\log_e\{x\sqrt{a} + (ax^2+b)^{\frac{1}{2}}\}$ if $a>0$ and $b>0$

$= C + \frac{1}{\sqrt{a}}\sinh^{-1}x\sqrt{\frac{a}{b}}$,, $a>0$,, $b>0$

$= C + \frac{1}{\sqrt{a}}\cosh^{-1}x\sqrt{\frac{-a}{b}}$,, $a>0$,, $b<0$

$= C + \frac{1}{\sqrt{(-a)}}\sin^{-1}x\sqrt{-\frac{a}{b}}$,, $a<0$,, $b>0$

Otherwise : put $x \equiv X\sqrt{\frac{b}{a}}$ or $x \equiv X\sqrt{\left(-\frac{b}{a}\right)}$, and use III. B. 3, 4, or 5.

7. $\int \frac{x\,dx}{(ax^2+b)^{\frac{1}{2}}} = C + \frac{1}{a}(ax^2+b)^{\frac{1}{2}}.$

8. $$\int \frac{dx}{x(ax^2+b)^{\frac{1}{2}}} = -\int \frac{dX}{(a+bX^2)^{\frac{1}{2}}} \text{ where } X \equiv \frac{1}{x}.$$ (See III. B. 6.)

Sp. Case: $$\int \frac{dx}{x(1+x^2)^{\frac{1}{2}}} = C - \sinh^{-1}\frac{1}{x} = C - \operatorname{cosech}^{-1} x$$

$$\int \frac{dx}{x(1-x^2)^{\frac{1}{2}}} = C - \cosh^{-1}\frac{1}{x} = C - \operatorname{sech}^{-1} x.$$

9. $$\int (Ax+B)(ax^2+b)^{\frac{1}{2}}dx = C + \left(\frac{A}{3}x^2 + \frac{B}{2}x + \frac{Ab}{3a}\right)(ax^2+b)^{\frac{1}{2}} + \frac{Bb}{2}\int \frac{dx}{(ax^2+b)^{\frac{1}{2}}}.$$ (See III. B. 6.)

Sp. Case: $$\int x(ax^2+b)^{\frac{1}{2}}dx = C + \frac{1}{3a}(ax^2+b)^{\frac{3}{2}}.$$

10. $$\int \frac{dx}{(2ax-x^2)^{\frac{1}{2}}} = C + \operatorname{vers}^{-1}\frac{x}{a} = C + \cos^{-1}\frac{a-x}{a}.$$

11. $$\int \frac{dx}{(2ax+x^2)^{\frac{1}{2}}} = C + \cosh^{-1}\frac{a+x}{a} = C + \log_e\{a+x+(2ax+x^2)^{\frac{1}{2}}\}.$$

12. $$\int \frac{dx}{x(2ax \pm x^2)^{\frac{1}{2}}} = C - \frac{(2ax \pm x^2)^{\frac{1}{2}}}{ax}.$$

13. $$\int \frac{dx}{(ax^2+bx+c)^{\frac{1}{2}}} = 2\sqrt{a}\int \frac{dX}{(4a^2X^2+4ac-b^2)^{\frac{1}{2}}}.$$

Where $X \equiv x + \frac{b}{2a}$. (See III. B. 6.)

14. $$\int \frac{(Ax+B)dx}{(ax^2+bx+c)^{\frac{1}{2}}} = \frac{A}{a}(ax^2+bx+c)^{\frac{1}{2}} + \frac{2Ba-Ab}{2a}\int \frac{dx}{(ax^2+bx+c)^{\frac{1}{2}}}.$$

(See II. B. 13.)

15. $$\int (Ax+B)(ax^2+bx+c)^{\frac{1}{2}}dx$$

$$= C + \left\{\frac{A}{3}x^2 + \left(\frac{B}{2} + \frac{Ab}{12a}\right)x + \frac{Bb}{4a} + \frac{Ac}{3a} - \frac{Ab^2}{8a^2}\right\}(ax^2+bx+c)^{\frac{1}{2}}$$

$$+ \left(\frac{Bc}{2} - \frac{Abc}{4a} - \frac{Bb^2}{8a} + \frac{Ab^3}{16a^2}\right)\int \frac{dx}{(ax^2+bx+c)^{\frac{1}{2}}}.$$

(See III. B. 13.)

16. $$\int \frac{(Ax+B)dx}{(ax^2+bx+c)^{\frac{3}{2}}} = C + 2\frac{bB - 2cA + (2aB-bA)x}{(4ac-b^2)(ax^2+bx+c)^{\frac{1}{2}}}.$$

17. $$\int \frac{Ax^{2n-1} + Bx^{2n-2} + \text{----} + K}{(ax^2 + bx + c)^{n+\frac{1}{2}}} dx = C + \frac{Lx^{2n-1} + Mx^{2n-2} + \text{----} + R}{(ax^2 + bx + c)^{n-\frac{1}{2}}}$$

if n is a positive integer; where L, M ---- R are constants to be determined by I. 11 (a).

18. $$\int \frac{Ax^m + Bx^{m-1} + \text{----} + K}{(ax^2 + bx + c)^{\frac{1}{2}}} dx$$

$$= (Px^{m-1} + Qx^{m-2} + \text{----} + S)(ax^2 + bx + c)^{\frac{1}{2}} + \int \frac{M dx}{(ax^2 + bx + c)^{\frac{1}{2}}}$$

where the constants P, Q, ---- S, M are determined by I. 11. For the last integral, see III. B. 13. Otherwise, see IX. A. 3.

APPENDIX N.—SECTION III. SOME SPECIAL AND SOME MORE GENERAL CASES.

A (4). Since $\dfrac{x}{x+b}=1-\dfrac{b}{x+b}$, $\therefore \displaystyle\int\frac{a}{(x+b)^2}=C+\frac{a}{b}\,\frac{x}{x+b}$.

A (8). $\displaystyle\int\frac{dx}{ax^2+b}=C+\frac{1}{\sqrt{ab}}\tan^{-1}\left(x\sqrt{\frac{a}{b}}\right)$.

A (9). $\displaystyle\int\frac{dx}{a^2-x^2}=C+\frac{1}{2a}\log\frac{a+x}{a-x}$.

A (12). When $n=2$, m may be 0 or 1; and the formula gives

$$\int\frac{dx}{1+x^2}=\tan^{-1}x \text{ and } \int\frac{xdx}{1+x^2}=\frac{1}{2}\log(1+x^2).$$

When $n=3$, m may be 0, or 1, or 2; and the formula gives

with $m=0$, $$\int\frac{dx}{1+x^3}=\frac{1}{3}\log(1+x)-\frac{1}{6}\log(x^2-x+1)$$
$$+\cdot577\tan^{-1}\frac{2x-1}{1\cdot732}+C,$$

and with $m=1$, $$\int\frac{xdx}{1+x^3}=-\frac{1}{3}\log(1+x)+\frac{1}{6}\log(x^2-x+1)$$
$$+\cdot577\tan^{-1}\frac{2x-1}{1\cdot732}+C,$$

and with $m=2$, $$\int\frac{x^2dx}{1+x^3}=\frac{1}{3}\log(1+x^3)+C.$$

When $n=4$, m may be 0, or 1, or 2, or 3; and the formula gives—

with $m=0$,

$$\int\frac{dx}{1+x^4}=\frac{1}{4\sqrt{2}}\log\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}+\frac{1}{2\sqrt{2}}\tan^{-1}\frac{x\sqrt{2}}{1-x^2}+C;$$

and with $m=1$,

$$\int\frac{xdx}{1+x^4}=C-\frac{1}{2}\tan^{-1}\frac{1}{x^2};$$

and with $m=2$,

$$\int\frac{x^2dx}{1+x^4}=\frac{1}{4\sqrt{2}}\log\frac{x^2-x\sqrt{2}+1}{x^2+x\sqrt{2}+1}+\frac{1}{2\sqrt{2}}\tan^{-1}\frac{x\sqrt{2}}{1-x^2}+C;$$

and with $m=3$,

$$\int\frac{x^3dx}{1+x^4}=\frac{1}{4}\log(1+x^4)+C.$$

B (2). $$\int \frac{x\,dx}{(1-2x)^{\frac{1}{2}}} = C - \frac{1+x}{3}(1-2x)^{\frac{1}{2}}.$$

$$\int \frac{x\,dx}{(1-ax)^{\frac{1}{2}}} = C - \frac{2(2+ax)}{3a^2}(1-ax)^{\frac{1}{2}}.$$

$$\int \frac{dx}{(x+1)\sqrt{x^2-1}} = C + \sqrt{\frac{x-1}{x+1}}.$$

$$\int \frac{dx}{(x-1)\sqrt{x^2-1}} = C - \sqrt{\frac{x+1}{x-1}}.$$

$$\int \sqrt{\frac{1+x}{1-x}}\,dx = C + \sin^{-1}x - \sqrt{1-x^2}.$$

$$\int \sqrt{\frac{x+a}{x+b}}\,dx = C + \sqrt{(x+a)(x+b)}$$
$$+ (a-b)\log\{\sqrt{x+a} + \sqrt{x+b}\}.$$

B (5). $$\int \frac{dx}{(b^2-a^2x^2)^{\frac{1}{2}}} = C + \frac{1}{a}\sin^{-1}\left(\frac{ax}{b}\right).$$

B (13). Another form applicable whether a be + or −:

$$\int \frac{dx}{(ax^2+bx+c)^{\frac{1}{2}}} = \int \frac{dX}{\left(aX^2 - \frac{b^2}{4a} + c\right)^{\frac{1}{2}}} \text{ with } X \equiv x + \frac{b}{2a}.$$

IV. LOGARITHMS AND EXPONENTIALS.

1. $\int e^x dx \quad = C + e^x$ ($e \equiv$ base of Neperian Logarithms).

2. $\int a^{nx} dx \quad = C + \frac{1}{n \log_e a} a^{nx}.$

Sp. Case: $\int e^{nx} dx = C + \frac{1}{n} e^{nx}.$

3. $\int \log_e x dx \quad = C + x \log_e x - x.$

4. $\int \log_b x dx \quad = C + x(\log_b x - \log_b e).$

Sp. Case: $b = 10$, $\int \log_{10} x dx = C + x(\log_{10} x - \cdot 43429 - - - -).$

5. $\int (\log_e x)^n dx = C + x\{(\log_e x)^n - n(\log_e x)^{n-1} + n(n-1)(\log_e x)^{n-2} - - - - - \pm n!\}.$

6. $\int x^m e^x dx \quad = C + e^x\{x^m - mx^{m-1} + m(m-1)x^{m-2} - - - - - - - - - \pm m!\}.$

7. $\int a^{\log_b x} dx \quad = \int x^{\log_b a} dx.$ (See III. A. 2.)

For other formulæ involving logarithms or exponentials, see VIII. 7–25 and IX. C.

Definite Integrals with Numerical (or Particular) Limits.

8. $\int_0^\infty e^{-ax^2} dx \quad = \frac{1}{2}\sqrt{\frac{\pi}{a}}$ where $a > 0$.

9. $\int_0^1 \left(\log \frac{1}{x}\right)^n dx = \int_0^\infty e^{-x} x^n dx = \Gamma(n+1).$ (See X. 1–6.)

Appendix O.

IV. (7). From the equality of the logarithms of the two sides, $a^{n \log_b x} = x^{n \log_b a}$ and $e^{n \log_e x} = x^n$.

Therefore

$$\int a^{n \log_b x} dx = \frac{x^{n \log_b a + 1}}{n \log_b a + 1}$$

and $\int e^{n \log_e x} dx = \frac{x^{n+1}}{n+1}.$

Also $\int_0^\infty e^{-ax} x^n dx = a^{-(n+1)} \Gamma(n+1).$

V.—HYPERBOLIC FUNCTIONS.

[$\text{Sinh}x \equiv \frac{1}{2}(e^x - e^{-x})$; $\cosh x \equiv \frac{1}{2}(e^x + e^{-x})$;

$\tanh x \equiv \frac{\sinh x}{\cosh x}$; $\coth x \equiv \frac{1}{\tanh x}$;

$\text{sech}\, x \equiv \frac{1}{\cosh x}$; $\text{cosech}\, x \equiv \frac{1}{\sinh x}$;

$\text{gd}x = \cos^{-1} \text{sech}\, x = \sin^{-1} \tanh x = \tan^{-1} \sinh x = 2 \tan^{-1} \tanh \frac{x}{2}$.]

1. $\int \sinh x.dx = C + \cosh x$.

2. $\int \cosh x.dx = C + \sinh x$.

3. $\int \tanh x.dx = C + \log_e \cosh x$.

4. $\int \coth x.dx = C + \log_e \sinh x$.

5. $\int \text{sech}\, x.dx = C + 2 \tan^{-1} e^x = C + \text{gd}x$.

6. $\int \text{cosech}\, x.dx = C + \log_e \frac{e^x - 1}{e^x + 1} = C + \log_e \tanh \frac{x}{2}$.

Note.—Every formula in VI. gives rise to a corresponding formula in hyperbolic functions, and *vice versâ*, by writing ix for x and using the identities:—

$$\sin ix = i \sinh x$$
$$\cos ix = \cosh x$$
$$\tan ix = i \tanh x$$

where $i \equiv \surd(-1)$.

VI.—TRIGONOMETRICAL FORMS.

Note.—The substitution of $px+q$ for x and of $p.dx$ for dx, gives more general forms of VI., 16, 19–25.

Sp. Case: $-x+\frac{\pi}{2}$ for x and $-dx$ for dx changes every trigonometrical ratio of x into its complementary function.

1. $\int \sin(px+q)dx = C - \frac{1}{p}\cos(px+q).$

$$\text{Sp. Case: } \int \sin x dx = C - \cos x.$$

2. $\int \cos(px+q)dx = C + \frac{1}{p}\sin(px+q).$

$$\text{Sp. Case: } \int \cos x dx = C + \sin x.$$

3. $\int \tan(px+q)dx = C - \frac{1}{p}\log_e \cos(px+q).$

$$\text{Sp. Case: } \int \tan x dx = C - \log_e \cos x.$$

4. $\int \cot(px+q)dx = C + \frac{1}{p}\log_e \sin(px+q).$

$$\text{Sp. Case: } \int \cot x dx = C + \log_e \sin x.$$

5. $\int \sec(px+q)dx = C + \frac{1}{2p}\log_e \frac{1+\sin(px+q)}{1-\sin(px+q)}$

$$= C + \frac{1}{p}\log_e \tan\left(\frac{\pi}{4}+\frac{px+q}{2}\right).$$

6. $\int \operatorname{cosec}(px+q)dx = C + \frac{1}{2p}\log_e \frac{1-\cos(px+q)}{1+\cos(px+q)}$

$$= C + \frac{1}{p}\log_e \tan\frac{px+q}{2}.$$

7. $\int \sin^2(px+q)dx = C + \frac{x}{2} - \frac{1}{2p}\sin(px+q)\cos(px+q).$

8. $\int \cos^2(px+q)dx = C + \frac{x}{2} + \frac{1}{2p}\sin(px+q)\cos(px+q).$

9. $\int \tan^2(px+q)dx \quad = C - x + \frac{1}{p}\tan(px+q).$

10. $\int \cot^2(px+q)dx \quad = C - x - \frac{1}{p}\cot(px+q).$

11. $\int \sec^2(px+q)dx \quad = C + \frac{1}{p}\tan(px+q).$

12. $\int \operatorname{cosec}^2(px+q)dx = C - \frac{1}{p}\cot(px+q).$

13. $\int \frac{\sin(px+q)dx}{\cos^2(px+q)} = C + \frac{1}{p}\sec(px+q).$

14. $\int \frac{\cos(px+q)dx}{\sin^2(px+q)} = C - \frac{1}{p}\operatorname{cosec}(px+q).$

15. $\int \frac{dx}{\sin(px+q)\cos(px+q)} = C + \frac{1}{p}\log_e\tan(px+q).$

16. $\int \sin^n x dx,\ \int \cos^n x dx,\ \int \frac{dx}{\sin^n x},\ \int \frac{dx}{\cos^n x},\ \int \tan^n x dx,\ \int \cot^n x dx.$

For the first two integrals use II. F. or IX. B. 1, 2.
,, ,, second pair ,, IX. B. 3, 4.
,, ,, last pair ,, VI. 21, 22.

See also VI. 19.

17. $\int \sin(px+q)\cos^n(px+q)dx = C - \frac{1}{(n+1)p}\cos^{n+1}(px+q).$

18. $\int \cos(px+q)\sin^n(px+q)dx = C + \frac{1}{(n+1)p}\sin^{n+1}(px+q).$

19. $\int \sin^m x \cos^n x dx.$ Four methods.

Method I. (1) if m is an odd positive integer, use $X \equiv \cos x$
(2) ,, n ,, ,, ,, ,, $X \equiv \sin x$
(3) ,, $m+n$,, even negative ,, ,, $X \equiv \tan x$

and the integrals become rational.

Method II. If m and n are positive integers, use II. F.

Method III. Use IX. B. 5, 6, 7 or 8.

Method IV. If m or n or both are fractional, use $X \equiv \sin x$ or $\equiv \cos x$, and expand the binomial factor which results.

20. $\int \sin^m x \cos^n x \sin qx \cos rx \text{ - - - - } dx.$

Where m, n, q, r - - - - are positive integers, use II. F.

21. $\int \tan^n x dx.$

If n is even, $= C + \dfrac{\tan^{n-1}x}{n-1} - \dfrac{\tan^{n-3}x}{n-3} + \text{ - - - - } \pm \tan x \mp x.$

„ „ odd $= C + \dfrac{\tan^{n-1}x}{n-1} - \dfrac{\tan^{n-3}x}{n-3} + \text{ - - - - } \pm \dfrac{\tan^2 x}{2} \pm \log_e \cos x.$

22. $\int \cot^n x dx = C - \dfrac{1}{n-1}\cot^{n-1}x + \dfrac{1}{n-3}\cot^{n-3}x \text{ - - - - - } \pm \cot x \mp x,$ if n even.

$= C - \dfrac{1}{n-1}\cot^{n-1}x + \dfrac{1}{n-3}\cot^{n-3}x \text{ - - - - - - } \pm \tfrac{1}{2}\cot^2 x$ $\pm \log_e \sin x$, if n odd.

23. $\int \dfrac{dx}{a + b\cos x} = C + \dfrac{1}{\sqrt{(b^2 - a^2)}} \cosh^{-1} \dfrac{a\cos x + b}{a + b\cos x}$ if $b^2 > a^2$

$= C + \dfrac{1}{\sqrt{(a^2 - b^2)}} \cos^{-1} \dfrac{a\cos x + b}{a + b\cos x}$ if $b^2 < a^2$.

24. $\int \dfrac{dx}{a + b\cos x + c\sin x} = \int \dfrac{dX}{a + \sqrt{(b^2 + c^2)}\cos X}$

where $X \equiv x - \tan^{-1}\dfrac{c}{b}$. (See VI. 23.)

Sp. Case: (1) when $c = 0$, it becomes VI. 23.

(2) when $b = 0$, „ „ $\int \dfrac{dx}{a + c\sin x}$.

25. $\int \dfrac{\cos^m x dx}{(a + b\cos x)^n} = \dfrac{1}{(a^2 - b^2)^{n-\frac{1}{2}}} \int (a\cos X - b)^m (a \quad b\cos X)^{n-m-1} dX,$

if $a > b$, $m + 1 < n$, where $\tan\dfrac{X}{2} \equiv \sqrt{\dfrac{a-b}{a+b}} \tan\dfrac{x}{2}$.

$= \dfrac{1}{(b^2 - a^2)^{n-\frac{1}{2}}} \int (b - a\cosh X)^m (b\cosh X - a)^{n-m+1} dX,$

if $a < b$, $m + 1 < n$, where $\tanh\dfrac{X}{2} \equiv \sqrt{\dfrac{b-a}{b+a}} . \tan\dfrac{x}{2}$.

Expand in each case by Binomial Theorem, and use II. A., then VI. 16 or 19, or Note at end of V.*

* See Appendix P.

Appendix P.

The definition of X here given is the best for ready calculation of its value; but it is well to note that

$$\tan\frac{X}{2} = \sqrt{\frac{a-b}{a+b}} \cdot \tan\frac{x}{2} \text{ means also}$$

$$\sin X = \sqrt{a^2-b^2}\frac{\sin x}{a+b\cos x}$$

$$\cos X = \frac{a\cos x+b}{a+b\cos x}$$

$$\tan X = \sqrt{a^2-b^2}\frac{\sin x}{a\cos x+b}$$

$$dX = \sqrt{a^2-b^2}\frac{dx}{a+b\cos x}.$$

Definite Integrals with Particular [or Numerical] Limits.

26. $$\int_0^{\pi/2}\sin^n x dx = \int_0^{\pi/2}\cos^n x dx = \frac{\Gamma(\frac{1}{2}).\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{n}{2}+1\right)}.$$ (See X. 1–6.)

27. $$\int_0^{\pi/2}\sin^m x\cos^n x dx = \frac{\Gamma\left(\frac{m+1}{2}\right).\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}.$$ " "

Appendix Q.—Some other Special and some more General Cases.

VI. (7) and (8).

$$\int \sin^2(px+q)dx = C + \frac{x}{2} - \frac{1}{4p}\sin 2(px+q).$$

$$\int \cos^2(px+q)dx = C + \frac{x}{2} + \frac{1}{4p}\sin 2(px+q).$$

VI. (17) and (18).

$$\int \sin(px+q)\cos(px+q)dx = C + \frac{1}{2p}\sin^2(px+q)$$
$$= C - \frac{1}{4p}\cos 2(px+q).$$

$$\int \sin(px+q)\cos(rx+k)dx = C - \frac{\cos\{(p+r)x+q+k\}}{2(p+r)} - \frac{\cos\{(p-r)x+q-k\}}{2(p-r)}.$$

$$\int \sin(px+q)\sin(rx+k)dx = C - \frac{\sin\{(p+r)x+q+k\}}{2(p+r)} + \frac{\sin\{(p-r)x+q-k\}}{2(p-r)}.$$

$$\int \cos(px+q)\cos(rx+k)dx = C + \frac{\sin\{(p+r)x+q+k\}}{2(p+r)} + \frac{\sin\{(p-r)x+q-k\}}{2(p-r)}.$$

$$\int \sin px \sin(px+q)dx = C + \frac{x}{2}\cos q - \frac{1}{4p}\sin(2px+q).$$

VI. (26) and (27).

$$\int_0^{\pi/2} \sin x dx = 1 = \int_0^{\pi/2} \cos x dx. \quad \int_0^{\pi} \sin x dx = 2.$$

$$\int_0^{\pi} \cos x dx = 0 = \int_0^{2\pi} \cos x dx = \int_0^{2\pi} \sin x dx.$$

$$\int_0^{\pi} \sin^2 x dx = \frac{\pi}{2} = \int_0^{\pi} \cos^2 x dx.$$

$$\int_0^{\pi/p} \sin^2(px+q)dx = \frac{\pi}{2p} = \int_0^{\pi/p} \cos^2(px+q)dx.$$

If $p = ld$ and $r = md$, l and m being *integers*: that is, if d be the greatest common factor or divisor of p and r, l and m being calculable by $\frac{l}{m} = \frac{p}{r}$: then $p \cdot \frac{2\pi}{d} = l \cdot 2\pi$ and $r \cdot \frac{2\pi}{d} = m \cdot 2\pi$. Therefore, since $2\pi = 360°$, the addition of $\frac{2\pi}{d}$ to x in any composite trigonometrical function of *both* $(px + q)$ *and* $(rx + k)$ brings the function recurrently back to the same value. It also does the same to any similar function of $\{(p+r)x + \text{constant}\}$ or of $\{(p-r)x + \text{constant}\}$. Therefore the definite integration between any value x_1 and $x_1 + \frac{2\pi}{d}$ of each of the three functions given above, VI. (17) and (18), namely of sin ( ) cos ( ), sin ( ) sin ( ), and cos ( ) cos ( ), gives zero integral.

Also $$\int_0^{\pi/p} \sin(px + q) \cos(px + q) dx = 0 .$$

If x denote flux of time, then $\frac{2\pi}{d}$ is the lapse of time, or "period," between successive recurrences of identical values of any composite trigonometrical function of $(px + q)$ and $(rx + k)$, corresponding to the "beats" of the composite harmonic function. The two harmonic functions have the different periods $\frac{2\pi}{p}$ and $\frac{2\pi}{r}$.

VII.—INVERSE FUNCTIONS.

[*Note.*—These can be transformed into integrals involving the corresponding direct functions by substitutions like $X \equiv \sin^{-1}x$ $\therefore x \equiv \sin X$ $\therefore dx \equiv \cos X dX$, etc.]

N.B.—$\sinh^{-1}x = \log_e\{x + \sqrt{(1+x^2)}\}$.

$$\cosh^{-1}x = \log_e\{x \pm \sqrt{(x^2-1)}\}\ ;\ x>1.$$

$$\tanh^{-1}x = \frac{1}{2}\log_e\frac{1+x}{1-x}\ ;\ x<1.$$

$$\coth^{-1}x = \frac{1}{2}\log_e\frac{x+1}{x-1}\ ;\ x>1.$$

$$\text{sech}^{-1}x = \cosh^{-1}\frac{1}{x} = \log_e\frac{1 \pm \sqrt{(1-x^2)}}{x}\ ;\ x<1.$$

$$\text{cosech}^{-1}x = \sinh^{-1}\frac{1}{x} = \log_e\frac{1+\sqrt{(1+x^2)}}{x},\ \text{if } x>0$$

$$= \log_e\frac{1-\sqrt{(1+x^2)}}{x},\ \text{if } x<0.$$

$$\text{gd}^{-1}x = \text{sech}^{-1}\cos x = \tanh^{-1}\sin x = \sinh^{-1}\tan x$$

$$= 2\tanh^{-1}\tan\frac{x}{2} = \log_e\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) = \frac{1}{2}\log_e\frac{1+\sin x}{1-\sin x}.$$

---

1. $\int \sin^{-1}x\,dx \quad = C + x\sin^{-1}x \pm (1-x^2)^{\frac{1}{2}}.$

2. $\int \cos^{-1}x\,dx \quad = C + x\cos^{-1}x \mp (1-x^2)^{\frac{1}{2}}$

$$= \int\left(\frac{\pi}{2} - \sin^{-1}x\right)dx.$$

3. $\int \tan^{-1}x\,dx \quad = C + x\tan^{-1}x - \frac{1}{2}\log_e(1+x^2).$

4. $\int \cot^{-1}x\,dx \quad = C + x\cot^{-1}x + \frac{1}{2}\log_e(1+x^2).$

5. $\int \sec^{-1}x\,dx \quad = C + x\sec^{-1}x - \log_e\{x + \sqrt{(x^2-1)}\}$

$$= C + x\sec^{-1}x - \cosh^{-1}x.$$

6. $\int \operatorname{cosec}^{-1}x\,dx = C + x\operatorname{cosec}^{-1}x + \log_e\{x + \sqrt{(x^2-1)}\}$
$= C + x\operatorname{cosec}^{-1}x + \cosh^{-1}x.$

7. $\int \sinh^{-1}x\,dx = C + x\sinh^{-1}x - \sqrt{(1+x^2)}.$

8. $\int \cosh^{-1}x\,dx = C + x\cosh^{-1}x - \sqrt{(x^2-1)}.$

9. $\int \tanh^{-1}x\,dx = C + x\tanh^{-1}x + \frac{1}{2}\log_e(1-x^2).$
$= C + \frac{1+x}{2}\log_e(1+x) + \frac{1-x}{2}\log_e(1-x).$

10. $\int \coth^{-1}x\,dx = C + x\coth^{-1}x + \frac{1}{2}\log_e(x^2-1)$
$= C + \frac{x+1}{2}\log_e(x+1) - \frac{x-1}{2}\log_e(x-1).$

11. $\int \operatorname{sech}^{-1}x\,dx = C + x\operatorname{sech}^{-1}x - \cos^{-1}x.$

12. $\int \operatorname{cosech}^{-1}x\,dx = C + x\operatorname{cosech}^{-1}x + \sinh^{-1}x.$

VIII. MIXED FUNCTIONS.

1. $\int \frac{\sin x dx}{x} = C + x - \frac{x^3}{3.3!} + \frac{x^5}{5.5!} - \frac{x^7}{7.7!} + \cdots.$

2. $\int \frac{\cos x dx}{x} = C + \log_e x - \frac{x^2}{2.2!} + \frac{x^4}{4.4!} - \frac{x^6}{6.6!} + \cdots.$

3. $\int \frac{\tan x dx}{x} = C + \frac{4(4-1)B_1 x^2}{2.2!} + \frac{4^2(4^2-1)B_2 x^4}{4.4!} + \frac{4^3(4^3-1)B_3 x^6}{6.6!} + \cdots.$*

4. $\int \frac{\cot x dx}{x} = C - \frac{1}{x} - \frac{4B_1 x}{1.2!} - \frac{4^2 B_2 x^3}{3.4!} - \frac{4^3 B_3 x^5}{5.6!} - \cdots.$*

5. $\int \frac{\sec x dx}{x} = \frac{1}{2x}\log\frac{1+\sin x}{1-\sin x} + \int \left\{\sin x + \frac{1}{3}\sin^3 x + \frac{1}{5}\sin^5 x + \cdots\right\}\frac{dx}{x^2}.$

Use II. A., II. F., and IX. C. 3 and 4.

6. $\int \frac{\operatorname{cosec} x}{x} dx = C - \frac{1}{x} + 2\left\{\frac{(2-1)B_1 x}{1.2!} + \frac{(2^3-1)B_2 x^3}{3.4!} + \frac{(2^5-1)B_3 x^5}{5.6!} + \cdots\right\}.$*

7. $\int \frac{e^x dx}{x} = C + \log_e x + x + \frac{x^2}{2.2!} + \frac{x^3}{3.3!} + \frac{x^4}{4.4!} + \cdots.$

8. $\int x^n e^{ax} dx$ [$n \equiv$ a positive integer]

$$= C + e^{ax}\left\{\frac{x^n}{a} - \frac{nx^{n-1}}{a^2} + \frac{n(n-1)}{a^3}x^{n-2} + \cdots \frac{n(n-1)(n-2)\cdots 2.1}{a^{n+1}}\right\}.$$

If n is not a positive integer, use II. D.

9. $\int \frac{e^x}{x^m} dx$ [$m \equiv$ a positive integer]

$$= C - e^x\left\{\frac{1}{(m-1)x^{m-1}} + \frac{1}{(m-1)(m-2)x^{m-2}} + \cdots + \frac{1}{(m-1)(m-2)\cdots 2.x}\right\} + \frac{1}{(m-1)!}\int \frac{e^x dx}{x}.$$ (See VIII. 7.)

Otherwise see IX. C. 7.

* For values of B_1 $B_2 \cdots$ see *note* at end of Section VIII.

10. $\int x^{m+nx}dx = \int x^m e^{nx\log_e x}dx$

$$= \int x^m \left\{ 1 + nx\log_e x + \frac{(nx\log_e x)^2}{2\,!} + \frac{(nx\log_e x)}{3\,!} + \cdots \right\} dx.$$

Use II. A. and VIII. 18.

11. $\int e^{ax+b}\sin(px+q)dx = C + \frac{e^{ax+b}\{a\sin(px+q) - p\cos(px+q)\}}{a^2+p^2.}.$

12. $\int e^{ax+b}\cos(px+q)dx = \int e^{ax+b}\sin(px+\overline{q+\tfrac{\pi}{2}})dx.$ (See VIII. 11.)

13. $\int x^n \cos^p x \sin^q x dx$, where p and q are positive integers.

By II. F. and II. A. reduce to IX. C., 1 and 2.

14. $\int F(x)f(\sin x, \cos x)dx$, [F( ) and f( ) ≡ rational integral functions].

By II. F. and II. A. reduce to IX. C., 1 and 2.

15. $\int \log_e(\sin mx)dx = C - \frac{x}{m}\log_e 2 - \frac{1}{2m}(\sin 2mx + \frac{1}{2^2}\sin 4mx$

$$+ \frac{1}{3^2}\sin 6mx + \frac{1}{4^2}\sin 8mx + \cdots)$$

$$\left[\text{if } o < mx < \frac{\pi}{2}\right].$$

16. $\int \log_e(\cos mx)dx = C - \frac{x}{m}\log_e 2 + \frac{1}{2m}(\sin 2mx - \frac{1}{2^2}\sin 4mx$

$$+ \frac{1}{3^2}\sin 6mx + \frac{1}{4^2}\sin 8mx - \cdots)$$

$$\left[\text{if } -\frac{\pi}{2} < mx < \frac{\pi}{2}\right].$$

17. $\int \log_e(\tan mx)dx = C - \frac{1}{m}(\sin 2mx + \frac{1}{3^2}\sin 6mx + \frac{1}{5^2}\sin 10mx$

$$+ \frac{1}{7^2}\sin 14mx + \cdots).$$

18. $\int x^m(\log_e x)^n dx = \int e^{(m+1)X}X^n dX$

$X \equiv \log_e x.$

(See VIII. 8.)

19. $\int x^m F(\log_a x)dx$, [F( ) ≡ a rational integral function]

$$= \int e^{(m+1)X} F\left(\frac{X}{\log_e a}\right) dX, \quad X \equiv \log_e x.$$

Use II. A. and VIII. 8.

Definite Integrals with Particular [or Numerical] Limits.

20. $\int_0^\infty e^{-x} x^n dx = \Gamma(n+1)$. (See X.)

21. $\int_0^1 x^m (\log x)^n dx = (-1)^n \frac{\Gamma(n+1)}{(m+1)^{n+1}}$. (See X.)

22. $\int_0^1 \frac{\log x}{1-x} dx = -\frac{\pi^2}{6}$.

23. $\int_0^\infty \frac{\sin px}{x} dx = \frac{\pi}{2}$ if $p > o$.

24. $\int_0^\infty \frac{\cos px}{x} dx = \infty$.

25. $\int_0^{\pi/2} \log_e(\sin x) dx = -\frac{\pi}{2} \log_e 2$.

Note.—Bernoulli's Numbers.

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10} etc., etc.
$\frac{1}{6}$	$\frac{1}{30}$	$\frac{1}{42}$	$\frac{1}{30}$	$\frac{5}{66}$	$\frac{691}{2730}$	$\frac{7}{6}$	$\frac{3617}{510}$	$\frac{43867}{798}$	$\frac{122277}{2310}$.

IX.—FORMULÆ OF REDUCTION.

Note.—Formulas of Reduction may be obtained by combinations of I. 4, I. 7, and II. A, B, C.

IX. A. *Algebraical.*

1. $\int x^m X^r dx$ where $X \equiv ax^n + b$

m, n, r being + or −, whole or fractional indices.

By the use of one of the subjoined formulas, the integral of any one of the following 9 functions may be reduced to that of any of the other 8:

$x^{m+n}X^{r+1}$	$x^m X^{r+1}$	$x^{m-n}X^{r+1}$
$x^{m+n}X^r$	$x^m X^r$	$x^{m-n}X^r$
$x^{m+n}X^{r-1}$	$x^m X^{r-1}$	$x^{m-n}X^{r-1}$.

(i) is useful when m and n are of opposite sign; (iii), (iv), (v) are useful when r is −;

(iii) and (vi) used together give the reduction from $x^m X^r$ to $x^m X^{r-2}$
(iv) ,, (vi) ,, ,, ,, ,, ,, ,, ,, ,, $x^{m-n}X^{r-2}$
(iv) ,, (viii) ,, ,, ,, ,, ,, ,, ,, ,, $x^{m-2n}X^{r-2}$.

N.B.—When r is a + integer, this integral can be dealt with by binomial expansion of X^r. In other particular cases the substitutions of III. A. 15 and III. A. 20 may be used.

(i) $$\int x^m X^r dx = \frac{1}{(m+1)b}\left\{ x^{m+1}X^{r+1} - (m+1+nr+n)a\int x^{m+n}X^r dx \right\}$$

(ii) $$= \frac{1}{(m+1+nr)a}\left\{ x^{m+1-n}X^{r+1} - (m+1-n)b\int x^{m-n}X^r dx \right\}$$

(iii) $$= \frac{1}{(nr+1)b}\left\{ -x^{m+1}X^{r+1} + (m+1+nr+n)\int x^m X^{r+1} dx \right\}$$

(iv) $$= \frac{x^{m+1}X^{r+2}}{(m+1)b^2} + \frac{a}{n(r+1)b^2}\left\{ x^{m+1+n}X^{r+1} - \frac{(m+1+nr+n)(m+1+nr+2n)}{m+1}\int x^{m+n}X^{r+1} dx \right\}$$

(v) $$= \frac{1}{n(r+1)a}\left\{ x^{m+1-n}X^{r+1} - (m+1-n)\int x^{m-n}X^{r+1} dx \right\}$$

(vi) $$= \frac{1}{m+1+nr}\left\{ x^{m+1}X^r + nrb\int x^m X^{r-1} dx \right\}$$

(vii) $\int x^m X^r dx = \frac{1}{m+1}\left\{ x^{m+1}X^r - nra\int x^{m+n}X^{r-1}dx \right\}$

(viii) $= \frac{1}{(m+1+nr)(m+1+nr-n)}\left\{ (m+1-n)x^{m+1}X^r + \frac{nr}{a}x^{m+1-n}X^{r+1} - (m+1-n)nr\frac{b^2}{a}\int x^{m-n}X^{r-1}dx \right\}$

2. $X \equiv (ax^{2n} + bx^n + c)$

(i) $\int x^m X^r dx = \frac{1}{m+1+nr}\left\{ x^{m+1}X^r + nrc\int x^m X^{r-1}dx - nra\int x^{m+2n}X^{r-1}dx \right\}$

(ii) $= \frac{1}{(m+1+2nr)a}\left\{ x^{m+1-2n}X^{r+1} - (m+1+nr-n)b\int x^{m-n}X^r dx - (m+1-2n)c\int x^{m-2n}X^r dx \right\}$

3. $X \equiv ax^2 + bx + c$

$\int x^m X^{-\frac{1}{2}}dx = \frac{1}{ma}\left\{ x^{m-1}X^{\frac{1}{2}} - (m-\tfrac{1}{2})b\int x^{m-1}X^{-\frac{1}{2}}dx - (m-1)c\int x^{m-2}X^{-\frac{1}{2}}dx \right\}$

If m be a + integer, this reduces to III. B. 14 and 13.

4. $X \equiv ax^2 + bx + c$

$\int X^{-r}dx = \frac{1}{(r-1)(4ac-b^2)}\left\{ (2ax+b)X^{-r+1} + 2(2r-3)a\int X^{-r+1}dx \right\}$

If r be a + integer, this reduces to III. A. 16, 17.
" " (+ integer + $\frac{1}{2}$) " " III. B. 13.

IX. B. *Trigonometrical.*

Note.—The following formulæ remain true when $px+q$ is substituted for x and pdx for dx. (See II. G.)

Sp. Case: If $p=-1$, and $q=\frac{\pi}{2}$ radians, in this substitution, we deduce a new formula in which each trigonometrical ratio is replaced by its complementary ratio.

N.B.—The following formulæ, when n and m are integers, reduce to the formulæ referred to in the right-hand column.

1. $\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$ } VI. 1, or III. A. 1.

2. $\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$ } VI. 2, or III. A. 1.

3. $\int \frac{dx}{\sin^n x} = \frac{-\cos x}{(n-1)\sin^{n-1}x} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2}x}$ } VI. 6, or 12.

4. $\int \frac{dx}{\cos^n x} = \frac{\sin x}{(n-1)\cos^{n-1}x} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2}x}$ } VI. 5, or 11.

5. $X_{m,n} \equiv \int \sin^m x\cos^n x\,dx.$

$$X_{m,n} = \frac{1}{m+n}\sin^{m+1}x\cos^{n-1}x + \frac{n-1}{m+n}X_{m,n-2}$$
$$= \frac{-1}{m+n}\sin^{m-1}x\cos^{n+1}x + \frac{m-1}{m+n}X_{m-2,n}$$

} VI. 1, 2, or 17, 18, or III. A. 1.

6. $X_{m,n} \equiv \int \frac{\sin^m x}{\cos^n x}dx.$

$$X_{m,n} = \frac{1}{n-1}\cdot\frac{\sin^{m+1}x}{\cos^{n-1}x} + \frac{n-m-2}{n-1}X_{m,n-2}$$
$$= -\frac{1}{m-n}\cdot\frac{\sin^{m-1}x}{\cos^{n-1}x} + \frac{m-1}{m-n}X_{m-2,n}$$

} VI. 1, 3, 5, or 16, or III. A. 1.

7. $X_{m,n} \equiv \int \frac{\cos^n x}{\sin^m x} dx$

$$X_{m,n} = -\frac{1}{m-1}\cdot\frac{\cos^{n+1}x}{\sin^{m-1}x} + \frac{m-n-2}{m-1}X_{m-2,n}$$

$$= \frac{1}{n-m}\cdot\frac{\cos^{n-1}x}{\sin^{n-1}x} + \frac{n-1}{n-m}X_{m,n-2}$$

VI. 1, 2, 4, or 16, or III. A. 1.

8. $X_{m,n} \equiv \int \frac{dx}{\sin^m x \cos^n x}$

$$X_{m,n} = \frac{1}{n-1}\cdot\frac{1}{\sin^{m-1}x\cos^{n-1}x} + \frac{n+m-2}{n-1}X_{m,n-2}$$

$$= -\frac{1}{m-1}\cdot\frac{1}{\sin^{m-1}x\cos^{n-1}x} + \frac{m+n-2}{m-1}X_{m-2,n}$$

VI. 5, 6, 15.

9. $\int \tan^n x dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x dx.$

IX. C. *Mixed Functions.*

1. $\int x^n \sin mx dx = -\frac{x^n}{m}\cos mx + \frac{n}{m}\int x^{n-1}\cos mx dx$

2. $\int x^n \cos mx dx = \frac{x^n}{m}\sin mx - \frac{n}{m}\int x^{n-1}\sin mx dx$

} VI. 1, or 2, when n is a positive integer. Otherwise, use II. D. and II. A.

3. $\int \frac{\sin mx.dx}{x^n} = -\frac{\sin mx}{(n-1)x^{n-1}} + \frac{m}{n-1}\int\frac{\cos mx dx}{x^{n-1}}$

4. $\int \frac{\cos mx}{x^n}dx = -\frac{\cos mx}{(n-1)x^{n-1}} - \frac{m}{n-1}\int\frac{\sin mx dx}{x^{n-1}}$

} If $n \equiv$ a positive integer VIII. 1 or 2. Otherwise, use II. D.

5. $\int (\log x)^n x^m dx = \frac{x^{m+1}}{m+1}(\log x)^n - \frac{n}{m+1}\int(\log x)^{n-1}x^m dx$ } III. A. 2 if $n \equiv$ a positive integer.

6. $\int x^m a^{nx} dx = \frac{x^m a^{nx}}{n\log_e a} - \frac{m}{n\log_e a}\int x^{m-1}a^{nx}dx$ } IV. 2 if $m \equiv$ a positive integer.

7. $\int \frac{e^x}{x^m}dx = \frac{-e^x}{(m-1)x^{m-1}} + \frac{1}{m-1}\int\frac{e^x dx}{x^{m-1}}$ } VIII. 7 if m is a whole number.

8. $\int e^{ax}\cos^n x dx = \frac{e^{ax}\cos^{n-1}x(n\sin x + a\cos x)}{n^2+a^2}$
$+ \frac{n(n-1)}{n^2+a^2}\int e^{ax}\cos^{n-2}x dx.$

} IV. 2 if $n \equiv$ even positive integer. VIII. 12 if $n \equiv$ odd positive integer.

9. $\int e^{ax}\sin^n x dx = \frac{e^{ax}\sin^{n-1}x(a\sin x - n\cos x)}{n^2+a^2}$
$+ \frac{n(n-1)}{n^2+a^2}\int e^{ax}\sin^{n-2}x dx$

} IV. 2 if n is an even positive integer. VIII. 11 if $n \equiv$ odd positive integer.

10. $\int \frac{e^{ax}}{\cos^n x}dx = -\frac{e^{ax}\{a\cos x - (n-2)\sin x\}}{(n-1)(n-2)\cos^{n-1}x} + \frac{a^2+(n-2)^2}{(n-1)(n-2)}\int\frac{e^{ax}dx}{\cos^{n-2}x}.$

11. $\int \frac{e^{ax}}{\sin^n x}dx = -\frac{e^{ax}\{a\sin x + (n-2)\cos x\}}{(n-1)(n-2)\sin^{n-1}x} + \frac{a^2+(n-2)^2}{(n-1)(n-2)}\int\frac{e^{ax}dx}{\sin^{n-2}x}.$

X.—GAMMA FUNCTIONS.

Properties of the Gamma Function.

1. Definition: $\Gamma(n) \equiv \int_0^\infty e^{-x}x^{n-1}dx \equiv \int_0^1 \left(\log_e \frac{1}{x}\right)^{n-1} dx$

where $n > o$.

2. $\Gamma(n+1) = n\Gamma(n)$.

3. If n is a positive integer $\Gamma(n) = (n-1)!$ and $\Gamma(1) = 1$.

4. $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$ if $n > o$ and < 1.

5. $\Gamma(n) = \underset{\mu=\infty}{\text{Lim}} \frac{\mu!\ \mu^n}{n(n+1)\cdots(n+\mu)}$ where μ is a positive integer.

N.B.—Legendre, in his *Traité des Fonctions Elliptiques*, Vol. II., gives an extended table of 1000 entries from $n=1$ to $n=2$, to 12 decimal places.

6. Table of Values of $1 + \log_{10}\Gamma(n)$.

n	0	1	2	3	4	5	6	7	8	9
1·0	·9999	·9753	·9513	·9280	·9053	·8834	·8621	·8415	·8215	·8021
1·1	·7834	·7653	·7478	·7310	·7147	·6990	·6839	·6694	·6554	·6421
1·2	·6292	·6170	·6052	·5940	·5834	·5732	·5636	·5545	·5459	·5378
1·3	·5302	·5231	·5165	·5104	·5047	·4995	·4948	·4905	·4868	·4834
1·4	·4805	·4781	·4761	·4745	·4734	·4726	·4724	·4725	·4731	·4740
1·5	·4755	·4772	·4794	·4820	·4850	·4884	·4921	·4963	·5008	·5057
1·6	·5110	·5167	·5227	·5291	·5359	·5430	·5505	·5583	·5665	·5750
1·7	·5839	·5931	·6027	·6126	·6229	·6335	·6444	·6556	·6672	·6790
1·8	·6913	·7038	·7167	·7299	·7433	·7571	·7712	·7856	·8004	·8154
1·9	·8307	·8463	·8622	·8784	·8949	·9117	·9288	·9462	·9638	·9818

N.B.—To calculate $\Gamma(n)$, when n is given:—

If $n \equiv$ a positive integer, see 3. If $n > 1$ and < 2, use the table.

If $n > 2$, by using (2) make the value depend on one in which n lies between 1 and 2. Thus $\Gamma(3{\cdot}52) = 2{\cdot}52 \times 1{\cdot}52\ \Gamma(1{\cdot}52)$.

If $n < 1$, use (2) to make the value depend on one in which $n > 1$: thus $\Gamma(n) = \frac{\Gamma(n+1)}{n}$. Example: $\Gamma({\cdot}63) = \frac{\Gamma(1{\cdot}63)}{{\cdot}63}$.

XI. DIFFERENTIAL EQUATIONS.

N.B.—$exp(X) \equiv e^X$.

XI. A. *First Order, First Degree.*

1. $X' + mX = 0, \quad X = Ce^{-mx}$, or $x = -\frac{1}{m}\log\frac{X}{C}$.

2. $X' + f(x) = 0, \quad X = C - \int f(x)dx$.

3. $X' + Xf(x) = 0, \quad X = C\, exp\{-\int f(x)dx\}$.

   Sp. Case: $f(x) = mx^n$; $\quad \log\frac{X}{C} = -\frac{m}{n+1}x^{n+1}$.

4. $X' + Xf(x) = \phi(x)$

   $X = exp\{-\int f(x)dx\}[C + \int \phi(x) exp\{\int f(x)dx\}dx]$.

   Sp. Case: $f(x) = k, \quad \therefore X' + kX = \phi(x)$;

   $X = e^{-kx}\{C + \int \phi(x)e^{kx}dx\}$.

5. $X' = \phi(x)f(X), \quad \int\frac{dX}{f(X)} = \int \phi(x)dx$.

   Sp. Case: $\phi(x) = 1, \quad \therefore X' = f(X), \quad x = \int\frac{dX}{f(X)}$.

6. $f(x,X)X' + \phi(x,X) = 0$. If the condition $\frac{\partial\phi}{\partial X} = \frac{\partial f}{\partial x}$ is fulfilled, then

$$\int \phi(x,X)dx + \int\left[f(x,X) - \int\frac{\partial\phi(x,X)}{\partial X}dx\right]dX = C$$

or

$$\int f(x,X)dX + \int\left[\phi(xX) - \int\frac{\partial f(x,X)}{\partial x}\partial X\right]dx = C$$

the integrations being partial.

Note.—If the equation as given does not fulfil the above condition, it may do so after being multiplied throughout by a function of x or X or both, called the Integrating Factor. (See Boole, chapters IV., V.) *E.g.* $(x^2X + X + 1) + X'(x + x^3)$.

Integrating Factor, $1/(1+x^2)$. Solution: $xX + \tan^{-1}x = C$.

7. $X'f(x,X) + \phi(x,X) = 0$ where $f(x,X) + \phi(x,X)$ is homogeneous in x,X. Substitute $X \equiv x\mathfrak{X}$ and reduce to XI. A. 5.

8. $(ax + bX + c)X' + (fx + gX + h) = 0$.
Assume $ax + bX + c \equiv z$; $fx + gX + h \equiv Z$, hence
$-Z'z + fz - gZ = 0$. (See XI. A. 7 or 3.)
Exc. when $a:b = f:g$, put $\mathfrak{X} \equiv ax + bX$, hence
$(\mathfrak{X} + c)(\mathfrak{X}' - a) + g\mathfrak{X} + bh = 0$. (See XI. A. 5 Sp. Case.)

9. $X' + Xf(x) = X^n\phi(x)$. Substitute $\mathfrak{X} \equiv X^{1-n}$. Then
$\mathfrak{X}' + (1 - n)\mathfrak{X}f(x) = (1 - n)\phi(x)$. (See XI. A. 4.)

10. If $xX' = (AX + B)(aX + b)$;

then $Cx^\alpha = \dfrac{AX + B}{aX + b}$, where $\alpha = Ab - aB$.

If $xX' = aX^2 + bX + c$;

then $Cx^\alpha = \dfrac{2aX + b - \alpha}{2aX + b + \alpha}$, where $\alpha = \sqrt{b^2 - 4ac}$.

11. If $xX' = (AX + B)(ax + b)$;
then $AX + B = x^{Ab} . e^{A(ax+C)}$.

IX. B. *First Order, Second or Higher Degree.*

1. $f(x,X,X')=0$. If possible, solve for X'. Each solution $X'=\phi(x,X)$, solved by XI. A. if possible, gives part of the general solution.

2. $f(X,X')=0$. Use XI. B. 1 if possible: otherwise solve for X if possible. Each solution $X=\phi(X')$ gives $x=\int\frac{\phi'(X')}{X'}dX'+C$. Then eliminate X' between the last two equations.

3. $f(x,X')=0$. Use XI. B. 1 if possible: otherwise solve for x if possible. Each solution $x=\phi(X')$ gives $X=\int X'\phi'(X')dX'+C$. Then eliminate X' between the last two equations.

4. $X=xX'+f(X')$. (Clairault's Equation) $X=cx+f(c)$.
See § 202, p. 123.

XI. C. *Second Order.*

1. $X'' + m^2X = 0, \quad X = A\cos mx + B\sin mx$
$= C\cos(mx + K).$

2. $X'' - m^2X = 0, \quad X = Ae^{mx} + Be^{-mx}.$

3. $X'' + aX' + bX = 0.$ Two forms :—

$$X = e^{-ax/2}\left\{ A\exp\left(\frac{x}{2}\sqrt{(a^2 - 4b)}\right) + B\exp\left(-\frac{x}{2}\sqrt{(a^2 - 4b)}\right)\right\}$$
when $a^2 > 4b$

$$\left.\begin{aligned} &= e^{-ax/2}\left\{ A\cos\left(\frac{x}{2}\sqrt{4b - a^2}\right) + B\sin\left(\frac{x}{2}\sqrt{4b - a^2}\right)\right\} \\ &= Ce^{-ax/2}\cos\left\{\frac{x}{2}\sqrt{(4b - a^2)} + K\right\} \end{aligned}\right\} \text{when } a^2 < 4b.$$

4. $X'' = f(x), \quad X = \iint f(x)dxdx + Ax + B.$

5. $X'' = f(X) \quad x = \int \frac{dX}{\sqrt{\{C + 2\int f(X)dX\}}} + K.$

6. $X'' + aX' + bX = f(x).$
By XI. C. 3, find $\mathfrak{X}$ by solving $\mathfrak{X}'' + a\mathfrak{X}' + b\mathfrak{X} = 0.$
" XI. A. 4, " Ξ " " $\mathfrak{X}\Xi' + \{2\mathfrak{X}' + a\mathfrak{X}\}\Xi = f(x).$
Then $X = \mathfrak{X}\int \Xi dx.$ See § 216, page 132.

7. $X'' + X'f(x) + XF(x) = \phi(x).$
Find $\mathfrak{X}$ if possible from $\mathfrak{X}'' + \mathfrak{X}'f(x) + \mathfrak{X}F(x) = 0.$
" Ξ by XI. A. from $\Xi'\mathfrak{X} + \Xi\{2\mathfrak{X}' + \mathfrak{X}f(x)\} = \phi(x).$
Then $X = \mathfrak{X}\int \Xi dx.$ See § 214, page 130.

8. $\frac{d^2X}{dx^2} = c^2\frac{d^2X}{dy^2}$, where X is a function of x and y:

General integral solution :—

$$X = f\left(x + \frac{y}{c}\right) + \phi\left(x - \frac{y}{c}\right)$$

where the forms of the functions $f(\ )$ and $\phi(\ )$ are determined by limiting conditions.

XI. D. *Order higher than Second.*

1. $f(x, X^{(n-1)}, X^{(n)})=0$. Put $\mathfrak{X}=X^{(n-1)}$, and equation becomes $f(x, \mathfrak{X}, \mathfrak{X}')=0$. (See XI. A, or B.)

2. $f(x, X^{(n-2)}, X^{(n-1)}, X^{(n)},)=0$. Put $\mathfrak{X}=X^{(n-2)}$ and equation becomes $f(x, \mathfrak{X}, \mathfrak{X}', \mathfrak{X}'')=0$. (See XI. C.)

3. $X^{(n)}=f(x)$.

$$X=\int^{(n)}f(x)dx^n+C_1x^{n-1}+C_2x^{n-2}+\text{----}+C_n.$$

4. $X^{(n)}+a_1X^{(n-1)}+a_2X^{(n-2)}+\text{-----}+a_{n-1}X'+a_nX=0$
$X=C_1e^{m_1x}+C_2e^{m_2x}+\text{----}+C_ne^{m_nx}$
where $m_1, m_2, \text{-----} m_n$ are the roots of the auxiliary equation $m^n+a_1m^{n-1}+a_2m^{n-2}+\text{----}+a_{n-1}m+a_n=0$.

Note 1.—If $m_1=p+q\sqrt{-1}$ and $m_2=p-q\sqrt{-1}$ are a pair of imaginary roots, the terms $C_1e^{m_1x}+C_2e^{m_2x}$ are equivalent to the *real* form e^{px} $(A\cos qx+B\sin qx)$.

Note 2.—If there be r *equal* roots $m_1, m_2\text{----}m_r$, each $=\mu$, the corresponding terms in the value of X are $(C_1+C_2x+C_3x^2+\text{----}+C^rx^{r-1})e^{\mu x}$. And if there are r pairs of imaginary roots each $=p\pm q\sqrt{-1}$, the terms are

$$e^{px}\{A_1\cos qx+B_1\sin qx+x(A_2\cos qx+B_2\sin qx)+\text{----}+x^{r-1}(A_r\cos qx+B_r\sin qx)\}$$

5. $X^{(n)}+a_1X^{(n-1)}+a_2X^{(n-2)}+\text{----}+a_nX=b_0+b_1x+b_2x^2+\text{----}+b_rx^r$.

Differentiate both sides $r+1$ times and solve the resulting equation by XI. D. 4. This solution is too general, having $n+r+1$ arbitrary constants: but by substituting in the equation and using I. 11, we get $r+1$ relations between the constants.

Otherwise: see XI. D. 6.

6. $X^{(n)} + a_1 X^{(n-1)} + a_2 X^{(n-2)} + \cdots + a_n X = f(x.)$

Let $X = F(x)$ be the solution on the supposition that $f(x) = 0$. (See XI. D. 4.)

$$\text{Then } X = F(x) + \sum_{r=1}^{r=n} A_r exp(m_r x) \int exp\{-m_r x f(x)\} dx,$$

where A_1 A_2 - - - - are such as to make the equation

$$\frac{A_1}{m - m_1} + \frac{A_2}{m - m_2} + \cdots + \frac{A_n}{m - m_n} = \frac{1}{m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_n}$$

identically true. (See II. E.)

Another method: By variation of Parameters; see Forsyth, § 75.

XI. E. *Partial Differential Equations.*

1. $\frac{\partial y}{\partial t} = a^2 \frac{\partial^2 y}{\partial x^2}$, $y = C exp(\alpha x + \alpha^2 a^2 t)$

where C and α are arbitrary constants,

or $y = \{A \cos \alpha x + B \sin \alpha x\} exp(-\alpha^2 a^2 t)$,

A and B being arbitrary constants.

General Solution : y = sum of any number of solutions like the above.

E.g. $y = \int_{\alpha_1}^{\alpha_2} F(\alpha) exp(\alpha x + \alpha^2 a^2 t) d\alpha$, where F is an arbitrary function.

2. $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$, $y = F(x + at) + f(x - at)$

where F and f denote arbitrary functions.

3. $a\frac{\partial^2 y}{\partial t^2} + b\frac{\partial^2 y}{\partial x \partial t} + c\frac{\partial^2 y}{\partial x^2} + f\frac{\partial y}{\partial t} + g\frac{\partial y}{\partial x} + hy = 0$

$y = Ce^{\alpha x + \beta t}$ where C, α, β are arbitrary; but α, β subject to the condition

$$a\beta^2 + b\alpha\beta + c\alpha^2 + f\beta + g\alpha + h = 0.$$

General Solution : y = the sum of any number of such particular solutions.

4. $a\frac{\partial y}{\partial t} + b\frac{\partial y}{\partial x} + c = 0$, $(ct + ay) = \phi(cx + by)$

where $\phi \equiv$ an arbitrary function.

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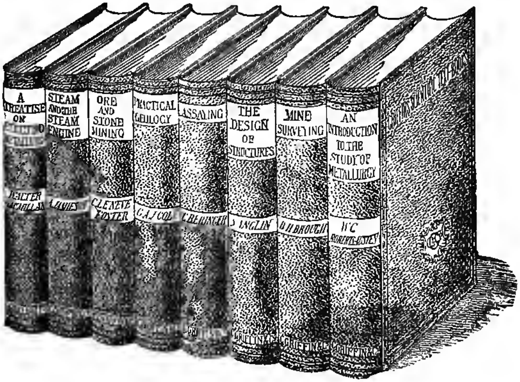

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